

Weighted multidegrees of polynomial automorphisms over a domain

Shigeru Kuroda *

Abstract

The notion of the weighted degree of a polynomial is a basic tool in Affine Algebraic Geometry. In this paper, we study the properties of the weighted multidegrees of polynomial automorphisms by a new approach which focuses on stable coordinates. We also present some applications of the generalized Shestakov-Umirbaev theory.

1 Introduction

Throughout this paper, k denotes an arbitrary domain unless otherwise stated. Let $k[\mathbf{x}] = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over k , where n is a positive integer. The automorphism group $\text{Aut}_k k[\mathbf{x}]$ of the k -algebra $k[\mathbf{x}]$ is a central object in Affine Algebraic Geometry. The purpose of this paper is to study the properties of the weighted multidegrees of elements of $\text{Aut}_k k[\mathbf{x}]$.

Let Γ be a *totally ordered additive group*, i.e., an additive group equipped with a total ordering such that $\alpha \leq \beta$ implies $\alpha + \gamma \leq \beta + \gamma$ for each $\alpha, \beta, \gamma \in \Gamma$. We denote $\Gamma_+ = \{\gamma \in \Gamma \mid \gamma > 0\}$ and $\Gamma_{\geq 0} = \{\gamma \in \Gamma \mid \gamma \geq 0\}$. Let $\mathbf{w} = (w_1, \dots, w_n)$ be an n -tuple of elements of Γ . We define the \mathbf{w} -*weighted Γ -grading*

$$k[\mathbf{x}] = \bigoplus_{\gamma \in \Gamma} k[\mathbf{x}]_{\gamma}$$

by setting $k[\mathbf{x}]_{\gamma}$ to be the k -submodule of $k[\mathbf{x}]$ generated by $x_1^{a_1} \cdots x_n^{a_n}$ for $a_1, \dots, a_n \in \mathbf{N}_0$ with $\sum_{i=1}^n a_i w_i = \gamma$ for each $\gamma \in \Gamma$. Here, \mathbf{N}_0 denotes the set of nonnegative integers. Write $f \in k[\mathbf{x}] \setminus \{0\}$ as $f = \sum_{\gamma \in \Gamma} f_{\gamma}$, where

2010 *Mathematical Subject Classification*. Primary 14R10; Secondary 13F20.

*Partly supported by the Grant-in-Aid for Young Scientists (B) 24740022, Japan Society for the Promotion of Science.

$f_\gamma \in k[\mathbf{x}]_\gamma$ for each $\gamma \in \Gamma$. Then, we define the **w-weighted degree** (**w-degree**, for short) of f by

$$\deg_{\mathbf{w}} f = \max\{\gamma \in \Gamma \mid f_\gamma \neq 0\}.$$

We define the **w-weighted initial form** (**w-initial form**, for short) of f by $f^{\mathbf{w}} = f_\delta$, where $\delta := \deg_{\mathbf{w}} f$. When $f = 0$, we define $f^{\mathbf{w}} = 0$ and $\deg_{\mathbf{w}} f = -\infty$. Here, $-\infty$ is a symbol which is less than any element of Γ . To denote elements of $\text{Aut}_k k[\mathbf{x}]$, we often use the notation $F = (f_1, \dots, f_n)$, $G = (g_1, \dots, g_n)$, etc, where each f_i and g_i represent the images of x_i by F and G , respectively. We define the **w-weighted degree** and **w-weighted multidegree** (**w-degree** and **w-multidegree**, for short) of F by

$$\deg_{\mathbf{w}} F = \sum_{i=1}^n \deg_{\mathbf{w}} f_i \quad \text{and} \quad \text{mdeg}_{\mathbf{w}} F = (\deg_{\mathbf{w}} f_1, \dots, \deg_{\mathbf{w}} f_n),$$

respectively. When $\Gamma = \mathbf{Z}$ and $\mathbf{w} = (1, \dots, 1)$, we denote “ $\deg_{\mathbf{w}}$ ” and “ $\text{mdeg}_{\mathbf{w}}$ ” simply by “ \deg ” and “ mdeg ”, respectively.

This paper consists of three parts. In the first part (Sections 2 through 5), we prove basic properties of the weighted degrees and multidegrees of elements of $\text{Aut}_k k[\mathbf{x}]$. Take any $F \in \text{Aut}_k k[\mathbf{x}]$ and $\emptyset \neq I \subset \{1, \dots, n\}$, and define J to be the set of $1 \leq j \leq n$ such that f_j belongs to $k[\{x_i \mid i \in I\}]$, and I_0 to be the set of $i_0 \in I$ such that $\deg_{\mathbf{w}} f_j$ belongs to $\sum_{i \in I \setminus \{i_0\}} \mathbf{N}_0 w_i$ for each $j \in J$. Here, for $N_i \subset \mathbf{Z}$ and $d_i \in \Gamma$ for $i = 1, \dots, r$ with $r \geq 1$, we define

$$N_1 d_1 + \dots + N_r d_r = \{a_1 d_1 + \dots + a_r d_r \mid a_i \in N_i \text{ for } i = 1, \dots, r\}.$$

We note that $J = \{1, \dots, n\}$ if $I = \{1, \dots, n\}$, and $I_0 = I$ if $J = \emptyset$.

With this notation, we have the following theorem.

Theorem 1.1. *Assume that $n \geq 1$ and k is a domain. Then, for any $\mathbf{w} \in \Gamma^n$, $F \in \text{Aut}_k k[\mathbf{x}]$ and $\emptyset \neq I \subset \{1, \dots, n\}$, the following assertions hold.*

- (i) *We have either (a) or (b) as follows:*
 - (a) *There exists a bijection $\sigma : J \rightarrow I$ such that $\deg_{\mathbf{w}} f_j = w_{\sigma(j)}$ for each $j \in J$.*
 - (b) *We have $\sum_{j \in J} \deg_{\mathbf{w}} f_j > \sum_{i \in I} w_i$ or $\#I > \#J$. For each $\mathbf{v} \in \Gamma^n$, there exists $i \in I_0$ such that x_i does not divide $(f_j^{\mathbf{w}})^{\mathbf{v}}$ for any $j \in J$.*
- (ii) *Assume that $\#I > \#J$. Then, for each $f \in k[\{f_j \mid j \in J\}] \setminus \{0\}$ and $\mathbf{v} \in \Gamma^n$, there exists $i \in I_0$ such that x_i does not divide $(f^{\mathbf{w}})^{\mathbf{v}}$.*

We prove Theorem 1.1 in Sections 4 and 5 with the aid of a recent result of the author [15].

As will be shown in Theorem 3.3 (i), we have

$$\deg_{\mathbf{w}} F \geq w_1 + \dots + w_n =: |\mathbf{w}|$$

for each $F \in \text{Aut}_k k[\mathbf{x}]$ and $\mathbf{w} \in \Gamma^n$. Detailed properties of the automorphisms satisfying $\deg_{\mathbf{w}} F = |\mathbf{w}|$ are given in Theorem 3.3 (ii). The following corollary is obtained by applying Theorem 1.1 (i) with $I = J = \{1, \dots, n\}$, since $\deg_{\mathbf{w}} F > |\mathbf{w}|$ implies (b), and hence implies $I_0 \neq \emptyset$.

Corollary 1.2. *Assume that $n \geq 1$ and k is a domain. Let $F \in \text{Aut}_k k[\mathbf{x}]$ and $\mathbf{w} \in \Gamma^n$ be such that $\deg_{\mathbf{w}} F > |\mathbf{w}|$. Then, there exists $1 \leq i \leq n$ such that $\deg_{\mathbf{w}} f_j$ belongs to $\sum_{l \neq i} \mathbf{N}_0 w_l$ for $j = 1, \dots, n$.*

We call $f \in k[\mathbf{x}]$ a *coordinate* of $k[\mathbf{x}]$ over k if $f = f_i$ for some $F \in \text{Aut}_k k[\mathbf{x}]$ and $1 \leq i \leq n$, and a *stable coordinate* of $k[\mathbf{x}]$ over k if f is a coordinate of $k[x_1, \dots, x_m]$ over k for some $m \geq n$ (cf. [18]). Clearly, a coordinate of $k[\mathbf{x}]$ over k is a stable coordinate of $k[\mathbf{x}]$ over k . However, the converse does not hold in general (cf. [2, Example 4.1]; see also [15, Section 3]).

In the situation of Theorem 1.1 (i), assume that $\#I \geq 2$. Then, for each $j \in J$, there exists $i \in I$ such that $\deg_{\mathbf{w}} f_j$ belongs to $\sum_{l \in I \setminus \{i\}} \mathbf{N}_0 w_l$ in both cases (a) and (b). From this observation, we see that the following theorem holds.

Theorem 1.3. *Assume that $n \geq 2$ and k is a domain. Let f be a stable coordinate of $k[\mathbf{x}]$ over k . Then, for each $\mathbf{w} \in \Gamma^n$, there exists $1 \leq i \leq n$ such that $\deg_{\mathbf{w}} f$ belongs to $\sum_{l \neq i} \mathbf{N}_0 w_l$.*

In fact, let $m \geq n$ and $F \in \text{Aut}_k k[x_1, \dots, x_m]$ be such that $f = f_1$, and J the set of $1 \leq j \leq m$ such that f_j belongs to $k[\mathbf{x}]$. Then, for each $j \in J$, there exists $i \in \{1, \dots, n\} =: I$ such that $\deg_{\mathbf{w}} f_j$ belongs to $\sum_{l \in I \setminus \{i\}} \mathbf{N}_0 w_l$ by the remark.

Next, let $C(\mathbf{w}, k)$ be the set of the \mathbf{w} -degrees of stable coordinates of $k[\mathbf{x}]$ over k , and let $C(\mathbf{w})$ be the set of $d \in \Gamma$ for which there exists $1 \leq i \leq n$ such that $d \geq w_i$ and $d = \sum_{j \neq i} a_j w_j$ for some $a_j \in \mathbf{N}_0$ for each $j \neq i$. Since

$$d = \deg_{\mathbf{w}} \left(x_i + \prod_{j \neq i} x_j^{a_j} \right)$$

holds for such d , we see that $C(\mathbf{w})$ is contained in $C(\mathbf{w}, k)$. It is clear that $\{w_1, \dots, w_n\}$ is contained in $C(\mathbf{w}, k)$. Therefore, $C(\mathbf{w}) \cup \{w_1, \dots, w_n\}$ is contained in $C(\mathbf{w}, k)$.

With the notation above, we have the following theorem.

Theorem 1.4. *Assume that $n \geq 1$ and k is a domain. Then, we have $C(\mathbf{w}, k) = C(\mathbf{w}) \cup \{w_1, \dots, w_n\}$ for any $\mathbf{w} \in (\Gamma_{\geq 0})^n$.*

We can derive Theorem 1.4 from Theorem 1.3 as follows. First, note that $\deg_{\mathbf{w}} f < w_j$ implies $f \in k[\{x_i \mid i \neq j\}]$ for each $f \in k[\mathbf{x}]$ and $1 \leq j \leq n$ by the choice of \mathbf{w} . Hence, f belongs to $k[\{x_i \mid i \in I\}]$, where $I := \{i \mid \deg_{\mathbf{w}} f \geq w_i\}$. Now, assume that f is a stable coordinate of $k[\mathbf{x}]$ over k . Then, f is a stable coordinate of $k[\{x_i \mid i \in I\}]$ over k . If $I = \{i\}$ for some $1 \leq i \leq n$, then f is a linear polynomial in x_i over k . Since $w_i \geq 0$, we have $\deg_{\mathbf{w}} f = w_i$. If $\#I \geq 2$, then we know by Theorem 1.3 that there exists $i \in I$ for which $\deg_{\mathbf{w}} f$ belongs to $\sum_{l \in I \setminus \{i\}} \mathbf{N}_0 w_l$, and hence to $\sum_{l \neq i} \mathbf{N}_0 w_l$. Since i is an element of I , we have $\deg_{\mathbf{w}} f \geq w_i$. Thus, $\deg_{\mathbf{w}} f$ belongs to $C(\mathbf{w})$. Therefore, $C(\mathbf{w}, k)$ is contained in $C(\mathbf{w}) \cup \{w_1, \dots, w_n\}$.

Next, we discuss tameness of automorphisms. Recall that $F \in \text{Aut}_k k[\mathbf{x}]$ is said to be *affine* if $\deg f_i = 1$ for $i = 1, \dots, n$, and *elementary* if there exist $1 \leq l \leq n$, $a \in k^\times$ and $p \in k[\{x_i \mid i \neq l\}]$ such that $f_l = ax_l + p$ and $f_i = x_i$ for each $i \neq l$. The subgroup $T_n(k)$ of $\text{Aut}_k k[\mathbf{x}]$ generated by all the affine automorphisms and elementary automorphisms of $k[\mathbf{x}]$ is called the *tame subgroup*. Then, the *Tame Generators Problem* asks whether every element of $\text{Aut}_k k[\mathbf{x}]$ is *tame*, i.e., belongs to $T_n(k)$. This is one of the difficult problems in Affine Algebraic Geometry. At present, it is known that the answer is affirmative if $n = 1$, or if $n = 2$ and k is a field by Jung [6] and van der Kulk [10], while negative if $n = 2$ and k is not a field by Nagata [21], or if $n = 3$ and k is of characteristic zero by Shestakov-Umirbaev [24].

For each subset S of $\text{Aut}_k k[\mathbf{x}]$ and $\mathbf{w} \in \Gamma^n$, we define

$$\text{mdeg}_{\mathbf{w}} S := \{\text{mdeg}_{\mathbf{w}} F \mid F \in S\}.$$

The following result is due to Karaś [7, Proposition 2.2], where \mathbf{N} denotes the set of positive integers throughout this paper.

Proposition 1.5 (Karaś). *Let $d_1, \dots, d_n \in \mathbf{N}$ be such that $d_1 \leq \dots \leq d_n$, where $n \geq 2$. If d_i belongs to $\sum_{j=1}^{i-1} \mathbf{N}_0 d_j$ for some $2 \leq i \leq n$, then (d_1, \dots, d_n) belongs to $\text{mdeg } T_n(\mathbf{C})$.*

The second part of this paper (Sections 6, 7 and 8) is aimed at generalizing this proposition. For this purpose, we introduce the following notation. Let κ be any commutative ring. Here, a “commutative ring” means one with a nonzero identity element. We remark that

$$\deg_{\mathbf{w}} fg = \deg_{\mathbf{w}} f + \deg_{\mathbf{w}} g \quad \text{and} \quad (fg)^{\mathbf{w}} = f^{\mathbf{w}} g^{\mathbf{w}} \quad (1.1)$$

hold for each $f, g \in \kappa[\mathbf{x}]$ and $\mathbf{w} \in \Gamma^n$ if $f^{\mathbf{w}}$ or $g^{\mathbf{w}}$ is a nonzero divisor of $\kappa[\mathbf{x}]$. Let $\text{Aut}_{\kappa}^{\mathbf{w}} \kappa[\mathbf{x}]$ be the set of $F \in \text{Aut}_{\kappa} \kappa[\mathbf{x}]$ such that $f_1^{\mathbf{w}}, \dots, f_n^{\mathbf{w}}$ are nonzero divisors of $\kappa[\mathbf{x}]$, let $E_n(\kappa)$ be the subgroup of $\text{Aut}_{\kappa} \kappa[\mathbf{x}]$ generated by all

the elementary automorphisms of $\kappa[\mathbf{x}]$, and let $E_n^{\mathbf{w}}(\kappa) = E_n(\kappa) \cap \text{Aut}_{\kappa}^{\mathbf{w}} \kappa[\mathbf{x}]$. Then, we define

$$|E_n^{\mathbf{w}}| := \bigcap_{\kappa} \text{mdeg}_{\mathbf{w}} E_n^{\mathbf{w}}(\kappa) = \bigcap_{m \in \mathbf{N}_0 \setminus \{1\}} \text{mdeg}_{\mathbf{w}} E_n^{\mathbf{w}}(\mathbf{Z}/m\mathbf{Z}),$$

where κ runs through all the commutative rings.

As mentioned later, every stable coordinate of $k[x_1, x_2]$ over k is a coordinate of $k[x_1, x_2]$ over k if k is an integrally closed domain (Theorem 7.1). Using this fact, we prove the following two theorems in Section 7.

Theorem 1.6. *Assume that $n = 3$ and k is a domain. Let $\mathbf{w} \in (\Gamma_+)^3$ and $(d_1, d_2, d_3) \in \text{mdeg}_{\mathbf{w}}(\text{Aut}_k k[\mathbf{x}])$ be such that at least two of d_1, d_2 and d_3 are not greater than $\max\{w_1, w_2, w_3\}$. Then, (d_1, d_2, d_3) belongs to $|E_3^{\mathbf{w}}|$.*

For each $w \in \Gamma^n$, and $F \in \text{Aut}_{\kappa} \kappa[\mathbf{x}]$ and $F' \in \text{Aut}_{\kappa'} \kappa'[\mathbf{x}]$ with κ and κ' any commutative rings, we define $F \sim_{\mathbf{w}} F'$ if $\text{mdeg}_{\mathbf{w}} F = \text{mdeg}_{\mathbf{w}} F'$. Then, Theorem 1.6 can be restated as follows: Let $F \in \text{Aut}_k k[\mathbf{x}]$ and $\mathbf{w} \in (\Gamma_+)^3$ be such that at least two of $\deg_{\mathbf{w}} f_1, \deg_{\mathbf{w}} f_2$ and $\deg_{\mathbf{w}} f_3$ are not greater than $\max\{w_1, w_2, w_3\}$. Then, for any commutative ring κ , there exists $G \in E_3^{\mathbf{w}}(\kappa)$ such that $G \sim_{\mathbf{w}} F$.

We note that $E_n(k) = T_n(k)$ when k is a field. In this case, we have the following theorem.

Theorem 1.7. *Assume that $n = 3$ and k is a field. If $F \in \text{Aut}_k k[\mathbf{x}]$ satisfies one of the following conditions for some $\mathbf{w} \in (\Gamma_+)^3$, then F belongs to $T_3(k)$:*
(1) $\deg_{\mathbf{w}} f_i \leq \max\{w_1, w_2, w_3\}$ for $i = 1, 2$.
(2) $\deg_{\mathbf{w}} f_2 - \max\{w_1, w_2, w_3\} < \deg_{\mathbf{w}} f_1 < \max\{w_1, w_2, w_3\}$.

In Section 8, we prove two kinds of sufficient conditions for elements of $\text{mdeg}_{\mathbf{w}}(\text{Aut}_k k[\mathbf{x}])$ to belong to $|E_n^{\mathbf{w}}|$ which can be viewed as generalizations of Proposition 1.5.

The third part of this paper (Section 9) is devoted to applications of the generalized Shestakov-Umirbaev theory. For $F \in \text{Aut}_k k[\mathbf{x}]$ and $\mathbf{w} \in \Gamma^n$, we say that F admits an *elementary reduction* for the weight \mathbf{w} if $\deg_{\mathbf{w}} F \circ E < \deg_{\mathbf{w}} F$ for some elementary automorphism E of $k[\mathbf{x}]$. Since $\deg_{\mathbf{w}} F \geq |\mathbf{w}|$ as mentioned, F admits no elementary reduction for the weight \mathbf{w} if $\deg_{\mathbf{w}} F = |\mathbf{w}|$.

Nagata [21] conjectured that a certain element of $\text{Aut}_k k[\mathbf{x}]$ for $n = 3$ does not belong to $T_3(k)$. Shestakov-Umirbaev solved this famous conjecture in the affirmative using the following criterion [24, Corollary 8].

Theorem 1.8 (Shestakov-Umirbaev). *Let k be a field of characteristic zero. If $\deg F > 3$ holds for $F \in T_3(k)$ with $f_3 = x_3$, then F admits an elementary reduction.*

Here, we simply say “elementary reduction” when $\Gamma = \mathbf{Z}$ and $\mathbf{w} = (1, \dots, 1)$. It is natural to ask whether a similar statement holds for general weights. We define $S(\mathbf{w}, k)$ to be the set of $F \in \text{Aut}_k k[\mathbf{x}]$ for $n = 3$ such that $\deg_{\mathbf{w}} F > |\mathbf{w}|$, and $f_3 = \alpha x_3 + p$ for some $\alpha \in k \setminus \{0\}$ and $p \in k[x_1, x_2]$ with $\deg_{\mathbf{w}} p \leq w_3$. By definition, we have $\deg_{\mathbf{w}} f_3 = w_3$ and $f_3^{\mathbf{w}} = \alpha x_3 + p'$ for such F , where $p' := p^{\mathbf{w}}$ if $\deg_{\mathbf{w}} p = w_3$, and $p' := 0$ otherwise.

Recently, the author [12], [13] generalized the Shestakov-Umirbaev theory. By means of this theory, we prove the following theorem in Section 9. This gives an affirmative answer to the question above.

Theorem 1.9. *Assume that k is a field of characteristic zero, and \mathbf{w} is an element of $(\Gamma_+)^3$. Then, every element of $S(\mathbf{w}, k) \cap T_3(k)$ admits an elementary reduction for the weight \mathbf{w} .*

The following theorem is also proved in Section 9. Part (i) of this theorem is a generalization of Proposition 1.5, while (ii) is a necessary condition for tameness of automorphisms obtained from Theorem 1.9.

Theorem 1.10. *Assume that $n = 3$ and k is a domain. Then, the following assertions hold for each $\mathbf{w} \in (\Gamma_+)^3$ and $F \in S(\mathbf{w}, k)$ with $\text{mdeg}_{\mathbf{w}} F = (d_1, d_2, d_3)$:*

- (i) *If d_i belongs to $\sum_{j \neq i} \mathbf{N}_0 d_j$ for some $1 \leq i \leq 3$, then there exists $G \in E_3^{\mathbf{w}}(\kappa)$ such that $g_3 = x_3$ and $\text{mdeg}_{\mathbf{w}} G = (d_1, d_2, d_3)$ for each commutative ring κ .*
- (ii) *If k is of characteristic zero and F belongs to $T_3(k)$, then d_i belongs to $\sum_{j \neq i} \mathbf{N}_0 d_j$ for some $1 \leq i \leq 3$.*

The author would like to thank Professors Amartya K. Dutta and Neena Gupta for helpful discussions on stable coordinates, and for pointing out that Theorem 7.1 is implicit in [1].

2 Initial principle

Throughout this section, let $n \in \mathbf{N}$ and $\mathbf{w} \in \Gamma^n$ be arbitrary. For given elements of $k[\mathbf{x}]$, we know what are the \mathbf{w} -degree and \mathbf{w} -initial form of their *product* thanks to (1.1), whereas those for the *sum* is unclear in general. The purpose of this section is to introduce basic techniques for treating the \mathbf{w} -degree and \mathbf{w} -initial form of the sum of polynomials.

The principle stated in the following lemma lies behind useful results proved in this and the next section. We omit the proof of this lemma, since the statement is obvious.

Lemma 2.1. For $(0, \dots, 0) \neq (f_1, \dots, f_l) \in k[\mathbf{x}]^l$ with $l \geq 1$, we set

$$\delta = \max\{\deg_{\mathbf{w}} f_i \mid i = 1, \dots, l\} \quad \text{and} \quad S = \{i \mid \deg_{\mathbf{w}} f_i = \delta\}.$$

Then, the following assertions hold:

- (i) $\deg_{\mathbf{w}} (f_1 + \dots + f_l) \leq \delta$.
- (ii) $\deg_{\mathbf{w}} (f_1 + \dots + f_l) = \delta$ if and only if $\sum_{i \in S} f_i^{\mathbf{w}} \neq 0$.
- (iii) If the equivalent conditions in (ii) are satisfied, then we have

$$(f_1 + \dots + f_l)^{\mathbf{w}} = \sum_{i \in S} f_i^{\mathbf{w}}.$$

For an r -tuple $F = (f_1, \dots, f_r)$ of elements of $k[\mathbf{x}]$ with $r \in \mathbf{N}$, we define the substitution map

$$k[x_1, \dots, x_r] \ni p(x_1, \dots, x_r) \mapsto p(f_1, \dots, f_r) \in k[\mathbf{x}].$$

As in the case of automorphisms, we denote this map by the same symbol F . When $f_i \neq 0$ for $i = 1, \dots, r$, we define

$$F^{\mathbf{w}} = (f_1^{\mathbf{w}}, \dots, f_r^{\mathbf{w}}) \quad \text{and} \quad \mathbf{w}_F = (\deg_{\mathbf{w}} f_1, \dots, \deg_{\mathbf{w}} f_r).$$

As a consequence of Lemma 2.1, we obtain the following proposition.

Proposition 2.2. For each $F \in (k[\mathbf{x}] \setminus \{0\})^r$ and $g \in k[x_1, \dots, x_r] \setminus \{0\}$, the following assertions hold:

- (i) $\deg_{\mathbf{w}} F(g) \leq \deg_{\mathbf{w}_F} g$.
- (ii) $\deg_{\mathbf{w}} F(g) = \deg_{\mathbf{w}_F} g$ if and only if $F^{\mathbf{w}}(g^{\mathbf{w}_F}) \neq 0$.
- (iii) If the equivalent conditions in (ii) are satisfied, then we have $F(g)^{\mathbf{w}} = F^{\mathbf{w}}(g^{\mathbf{w}_F})$.

Proof. Write $g = \sum_{i_1, \dots, i_r} a_{i_1, \dots, i_r} x_1^{i_1} \cdots x_r^{i_r}$ with $a_{i_1, \dots, i_r} \in k$, and set

$$q_i = a_{i_1, \dots, i_r} x_1^{i_1} \cdots x_r^{i_r} \quad \text{and} \quad p_i = a_{i_1, \dots, i_r} f_1^{i_1} \cdots f_r^{i_r}$$

for each $i = (i_1, \dots, i_r)$. Then, we have $g = \sum_i q_i$ and $F(g) = \sum_i p_i$. Define $\delta = \max\{\deg_{\mathbf{w}} p_i \mid i\}$ and $S = \{i \mid \deg_{\mathbf{w}} p_i = \delta\}$. By applying Lemma 2.1 to $(p_i)_i$, we obtain the following statements:

- (i') $\deg_{\mathbf{w}} F(g) \leq \delta$.
- (ii') $\deg_{\mathbf{w}} F(g) = \delta$ if and only if $h := \sum_{i \in S} p_i^{\mathbf{w}}$ is nonzero.
- (iii') If the equivalent conditions in (ii') are satisfied, then we have $F(g)^{\mathbf{w}} = h$.

Hence, it suffices to show that $\deg_{\mathbf{w}_F} g = \delta$ and $F^{\mathbf{w}}(g^{\mathbf{w}_F}) = h$. Note that

$$\deg_{\mathbf{w}} p_i = \sum_{l=1}^r i_l \deg_{\mathbf{w}} f_l = \sum_{l=1}^r i_l \deg_{\mathbf{w}_F} x_l = \deg_{\mathbf{w}_F} q_i \quad (2.1)$$

$$F^{\mathbf{w}}(q_i) = a_{i_1, \dots, i_r} (f_1^{\mathbf{w}})^{i_1} \cdots (f_r^{\mathbf{w}})^{i_r} = (a_{i_1, \dots, i_r} f_1^{i_1} \cdots f_r^{i_r})^{\mathbf{w}} = p_i^{\mathbf{w}} \quad (2.2)$$

for each $i = (i_1, \dots, i_r)$ with $a_{i_1, \dots, i_r} \neq 0$. Hence, we have

$$\deg_{\mathbf{w}_F} g = \max\{\deg_{\mathbf{w}_F} q_i \mid i\} = \max\{\deg_{\mathbf{w}} p_i \mid i\} = \delta$$

by (2.1). Thus, i belongs to S if and only if $\deg_{\mathbf{w}_F} q_i = \deg_{\mathbf{w}_F} g$. This implies that $g^{\mathbf{w}_F} = \sum_{i \in S} q_i$. Therefore, we conclude that

$$F^{\mathbf{w}}(g^{\mathbf{w}_F}) = F^{\mathbf{w}}\left(\sum_{i \in S} q_i\right) = \sum_{i \in S} F^{\mathbf{w}}(q_i) = \sum_{i \in S} p_i^{\mathbf{w}} = h$$

by (2.2). □

For each k -subalgebra A of $k[\mathbf{x}]$ and $\mathbf{w} \in \Gamma^n$, we define $A^{\mathbf{w}}$ to be the k -submodule of $k[\mathbf{x}]$ generated by $\{f^{\mathbf{w}} \mid f \in A\}$. In view of (1.1), we see that $A^{\mathbf{w}}$ is a k -subalgebra of $k[\mathbf{x}]$. We call $A^{\mathbf{w}}$ the *initial algebra* of A for the weight \mathbf{w} . For $g_1, \dots, g_l \in k[\mathbf{x}]$, it is clear that

$$k[g_1, \dots, g_l]^{\mathbf{w}} \supset k[g_1^{\mathbf{w}}, \dots, g_l^{\mathbf{w}}],$$

but the equality does not hold in general. We mention that the k -algebra $A^{\mathbf{w}}$ is not always finitely generated even if A is finitely generated (see e.g. [11]).

We note that f_1, \dots, f_r are algebraically independent over k if and only if the substitution map $F : k[x_1, \dots, x_r] \rightarrow k[\mathbf{x}]$ is injective. The following corollary is a consequence of Proposition 2.2.

Corollary 2.3. *Let $F \in (k[\mathbf{x}] \setminus \{0\})^r$ be such that $F^{\mathbf{w}}$ is injective. Then, the following assertions hold:*

- (i) $\deg_{\mathbf{w}} F(g) = \deg_{\mathbf{w}_F} g$ and $F(g)^{\mathbf{w}} = F^{\mathbf{w}}(g^{\mathbf{w}_F})$ hold for each $g \in k[x_1, \dots, x_r]$.
- (ii) F is injective.
- (iii) $k[f_1, \dots, f_r]^{\mathbf{w}} = k[f_1^{\mathbf{w}}, \dots, f_r^{\mathbf{w}}]$.

Proof. (i) The assertion is obvious if $g = 0$. So assume that $g \neq 0$. Then, we have $g^{\mathbf{w}_F} \neq 0$, and so $F^{\mathbf{w}}(g^{\mathbf{w}_F}) \neq 0$ by the injectivity of $F^{\mathbf{w}}$. Hence, we get $\deg_{\mathbf{w}} F(g) = \deg_{\mathbf{w}_F} g$ and $F(g)^{\mathbf{w}} = F^{\mathbf{w}}(g^{\mathbf{w}_F})$ by Proposition 2.2 (ii) and (iii).

(ii) If $F(g) = 0$ for $g \in k[x_1, \dots, x_r]$, then we have $\deg_{\mathbf{w}_F} g = \deg_{\mathbf{w}} F(g) = -\infty$ by (i). This implies that $g = 0$. Therefore, F is injective.

(iii) “ \supset ” is clear as mentioned above. To show “ \subset ”, it suffices to check that $f^{\mathbf{w}}$ belongs to $k[f_1^{\mathbf{w}}, \dots, f_r^{\mathbf{w}}]$ for each $f \in k[f_1, \dots, f_r]$. Let g be an element of $k[x_1, \dots, x_r]$ such that $f = F(g)$. Then, $f^{\mathbf{w}}$ is equal to $F^{\mathbf{w}}(g^{\mathbf{w}_F})$ by (i), and hence belongs to $k[f_1^{\mathbf{w}}, \dots, f_r^{\mathbf{w}}]$. This proves “ \subset ”. \square

We remark that, if w_1, \dots, w_n are linearly independent over \mathbf{Z} , then $f^{\mathbf{w}}$ is a monomial for each $f \in k[\mathbf{x}] \setminus \{0\}$, since distinct monomials have distinct \mathbf{w} -degrees. Hence, we have the following corollary to Proposition 2.2.

Corollary 2.4. *If $\deg_{\mathbf{w}} f_1, \dots, \deg_{\mathbf{w}} f_r$ are linearly independent over \mathbf{Z} for $F \in (k[\mathbf{x}] \setminus \{0\})^r$, then $F^{\mathbf{w}}$ is injective.*

Proof. Put $G = F^{\mathbf{w}}$. Take any $p \in k[x_1, \dots, x_r] \setminus \{0\}$. Then, $p^{\mathbf{w}_G}$ is a monomial by the remark, since $\mathbf{w}_G = (\deg_{\mathbf{w}} f_1, \dots, \deg_{\mathbf{w}} f_r)$. Since $G^{\mathbf{w}}(x_i) = (f_i^{\mathbf{w}})^{\mathbf{w}} \neq 0$ for each i , it follows that $G^{\mathbf{w}}(p^{\mathbf{w}_G}) \neq 0$. Thus, we get $G(p)^{\mathbf{w}} = G^{\mathbf{w}}(p^{\mathbf{w}_G}) \neq 0$ by Proposition 2.2 (iii). This implies that $G(p) \neq 0$. Therefore, G is injective. \square

3 Degrees of polynomial automorphisms

Throughout this section, let $n \in \mathbf{N}$ be arbitrary. We prove basic properties of the weighted degrees and multidegrees of elements of $\text{Aut}_k k[\mathbf{x}]$.

Lemma 3.1. *Let $F \in \text{Aut}_k k[\mathbf{x}]$ and $\mathbf{w} \in \Gamma^n$ be such that*

$$\deg_{\mathbf{w}} f_1 \leq \dots \leq \deg_{\mathbf{w}} f_n \quad \text{and} \quad w_1 \leq \dots \leq w_n. \quad (3.1)$$

Then, the following assertions hold:

- (i) *If $\deg_{\mathbf{w}} f_i < w_j$ for $i, j \in \{1, \dots, n\}$, then we have $i < j$.*
- (ii) *Assume that $w_1 \geq 0$ and let $1 \leq i < n$ be an integer. If $\deg_{\mathbf{w}} f_i < w_{i+1}$, then $k[f_1, \dots, f_i] = k[x_1, \dots, x_i]$. If furthermore $\deg_{\mathbf{w}} f_{i+1} < w_{i+2}$ or $i + 1 = n$, then $f_{i+1} = \alpha x_{i+1} + p$ for some $\alpha \in k^\times$ and $p \in k[x_1, \dots, x_i]$.*
- (iii) *Assume that $w_1 > 0$. Let $i, j \in \{1, \dots, n\}$ be such that $\deg_{\mathbf{w}} f_i = w_j$. Set $j_0 = \min\{l \mid w_l = w_j\}$ and $j_1 = \max\{l \mid w_l = w_j\}$. Then, we have*

$$f_i = g + a_{j_0} x_{j_0} + \dots + a_{j_1} x_{j_1}$$

for some $g \in k[x_1, \dots, x_{j_0-1}]$ and $a_{j_0}, \dots, a_{j_1} \in k$.

Proof. (i) Let f'_l be the linear part of f_l for each l . Then, the Jacobian of (f'_1, \dots, f'_n) is equal to that of F , and hence is an element of k^\times . Thus, f'_1, \dots, f'_n are linearly independent over k . Note that $\deg_{\mathbf{w}} f'_l \leq \deg_{\mathbf{w}} f_l$ for each l . Since $\deg_{\mathbf{w}} f_i < w_j$ by assumption, it follows that $\deg_{\mathbf{w}} f'_{i'} < w_{j'}$

for each $i' \leq i$ and $j' \geq j$ by (3.1). Thus, f'_1, \dots, f'_i belong to the k -module $kx_1 + \dots + kx_{j-1}$. Since f'_1, \dots, f'_i are linearly independent over k , we conclude that $i \leq j - 1 < j$.

(ii) Since $\deg_{\mathbf{w}} f_i < w_{i+1}$ by assumption, we have $\deg_{\mathbf{w}} f_{i'} < w_{j'}$ for each $i' \leq i$ and $j' \geq i + 1$ by (3.1). Since w_l 's are nonnegative, it follows that f_1, \dots, f_i belong to $k[\mathbf{x}_0] := k[x_1, \dots, x_i]$. This implies that $k[f_1, \dots, f_i] = k[\mathbf{x}_0]$. Next, assume that $\deg_{\mathbf{w}} f_{i+1} < w_{i+2}$ or $i + 1 = n$. Then, we have $k[\mathbf{x}_0][x_{i+1}] = k[f_1, \dots, f_{i+1}]$ similarly. Since $k[f_1, \dots, f_i] = k[\mathbf{x}_0]$, it follows that $k[\mathbf{x}_0][x_{i+1}] = k[\mathbf{x}_0][f_{i+1}]$. Therefore, f_{i+1} has the required form.

(iii) By the maximality of j_1 , we have $\deg_{\mathbf{w}} f_i = w_j < w_{j_1+1}$ or $j_1 = n$. Since w_l 's are positive, f_i belongs to $k[x_1, \dots, x_{j_1}]$ in either case. Write $f_i = g + h$, where $g \in k[x_1, \dots, x_{j_0-1}]$ and $h \in \sum_{l=j_0}^{j_1} x_l k[x_1, \dots, x_{j_1}]$. It remains only to show that $\deg h = 1$. Let $x_{j'}m$ be any monomial appearing in h , where $j_0 \leq j' \leq j_1$ and $m \in k[x_1, \dots, x_{j_1}]$. Then, we have $\deg_{\mathbf{w}} x_{j'}m \leq \deg_{\mathbf{w}} h \leq \deg_{\mathbf{w}} f_i = w_j$. Since

$$\deg_{\mathbf{w}} x_{j'}m = \deg_{\mathbf{w}} x_{j'} + \deg_{\mathbf{w}} m \geq \deg_{\mathbf{w}} x_{j'} = w_{j'} = w_j,$$

it follows that $\deg_{\mathbf{w}} m = 0$. This implies that m belongs to $k \setminus \{0\}$ by the positivity of w_l 's. Therefore, we conclude that $\deg h = 1$. \square

We say that $F \in \text{Aut}_k k[\mathbf{x}]$ is *triangular* if f_i belongs to $k[x_1, \dots, x_i]$ for $i = 1, \dots, n$. The following proposition can be proved similarly to Lemma 3.1 (ii).

Proposition 3.2. *Assume that $F \in \text{Aut}_k k[\mathbf{x}]$ and $\mathbf{w} \in (\Gamma_{\geq 0})^n$ satisfy (3.1). If $\deg_{\mathbf{w}} f_i < w_{i+1}$ for $i = 1, \dots, n - 1$, then F is triangular.*

In the study of polynomial automorphisms, the notion of the \mathbf{w} -degree of a differential form is important. Let $\Omega_{k[\mathbf{x}]/k}$ be the module of differentials of $k[\mathbf{x}]$ over k , and ω an element of the r -th exterior power $\bigwedge^r \Omega_{k[\mathbf{x}]/k}$ of the $k[\mathbf{x}]$ -module $\Omega_{k[\mathbf{x}]/k}$ for $r \in \mathbf{N}$. Then, we can uniquely write

$$\omega = \sum_{1 \leq i_1 < \dots < i_r \leq n} f_{i_1, \dots, i_r} dx_{i_1} \wedge \dots \wedge dx_{i_r},$$

where $f_{i_1, \dots, i_r} \in k[\mathbf{x}]$ for each i_1, \dots, i_r . Here, df denotes the differential of f for each $f \in k[\mathbf{x}]$. We define the \mathbf{w} -degree of ω by

$$\deg_{\mathbf{w}} \omega = \max\{\deg_{\mathbf{w}}(f_{i_1, \dots, i_r} x_{i_1} \cdots x_{i_r}) \mid 1 \leq i_1 < \dots < i_r \leq n\}.$$

Let f_1, \dots, f_r be elements of $k[\mathbf{x}] \setminus \{0\}$. Then, $df_1 \wedge \dots \wedge df_r \neq 0$ implies that f_1, \dots, f_r are algebraically independent over k (cf. [19, Section 26]). By

definition, we have

$$\begin{aligned}
& \deg_{\mathbf{w}} df_1 \wedge \cdots \wedge df_r \\
&= \max \left\{ \deg_{\mathbf{w}} \left(\left| \frac{\partial(f_1, \dots, f_r)}{\partial(x_{i_1}, \dots, x_{i_r})} \right| x_{i_1} \cdots x_{i_r} \right) \mid 1 \leq i_1 < \cdots < i_r \leq n \right\} \\
&\leq \sum_{i=1}^r \deg_{\mathbf{w}} f_i,
\end{aligned} \tag{3.2}$$

in which the equality holds if and only if $df_1^{\mathbf{w}} \wedge \cdots \wedge df_r^{\mathbf{w}} \neq 0$.

Now, let \mathfrak{S}_n be the symmetric group of $\{1, \dots, n\}$. Then,

$$\mathbf{w}_{\sigma} := (w_{\sigma(1)}, \dots, w_{\sigma(n)})$$

belongs to $|E_n^{\mathbf{w}}|$ for each $\sigma \in \mathfrak{S}_n$. Hence, $\text{mdeg}_{\mathbf{w}} F$ belongs to $|E_n^{\mathbf{w}}|$ for each $F \in \text{Aut}_k k[\mathbf{x}]$ with $\deg_{\mathbf{w}} F = |\mathbf{w}|$ by (ii) of the following proposition.

Theorem 3.3. *For each $F \in \text{Aut}_k k[\mathbf{x}]$ and $\mathbf{w} \in \Gamma^n$, the following assertions hold:*

- (i) *There exists $\sigma \in \mathfrak{S}_n$ such that $\deg_{\mathbf{w}} f_i \geq w_{\sigma(i)}$ for $i = 1, \dots, n$. Hence, we have $\deg_{\mathbf{w}} F \geq |\mathbf{w}|$.*
- (ii) *The following conditions are equivalent:*
 - (a) $\text{mdeg}_{\mathbf{w}} F = \mathbf{w}_{\sigma}$ for some $\sigma \in \mathfrak{S}_n$;
 - (b) $\deg_{\mathbf{w}} F = |\mathbf{w}|$;
 - (c) $F^{\mathbf{w}}$ is injective, i.e., $f_1^{\mathbf{w}}, \dots, f_n^{\mathbf{w}}$ are algebraically independent over k ;
 - (d) $F^{\mathbf{w}}$ belongs to $\text{Aut}_k k[\mathbf{x}]$.
- (iii) *If $\text{mdeg}_{\mathbf{w}} F = \mathbf{w}$, then we have $\text{mdeg}_{\mathbf{w}} F^{-1} = \mathbf{w}$.*

Proof. (i) Let $\tau, \rho \in \mathfrak{S}_n$ be such that

$$\deg_{\mathbf{w}} f_{\tau(1)} \leq \cdots \leq \deg_{\mathbf{w}} f_{\tau(n)} \quad \text{and} \quad w_{\rho(1)} \leq \cdots \leq w_{\rho(n)}.$$

Then, we have $\deg_{\mathbf{w}} f_{\tau(i)} \geq w_{\rho(i)}$ for each i by Lemma 3.1 (i). Put $\sigma = \rho \circ \tau^{-1}$. Then, $\deg_{\mathbf{w}} f_i \geq w_{\sigma(i)}$ holds for each i . The last statement is clear.

(ii) Clearly, (a) implies (b). By (i), we see that (b) implies (a). So we show that (b), (c) and (d) are equivalent. Let JF be the Jacobi matrix of F . Then, $\det JF$ belongs to k^{\times} . Hence, we know by (3.2) that

$$\deg_{\mathbf{w}} F \geq \deg_{\mathbf{w}} df_1 \wedge \cdots \wedge df_n = \deg_{\mathbf{w}} (\det JF) dx_1 \wedge \cdots \wedge dx_n = |\mathbf{w}|,$$

in which the equality holds if and only if $\eta := df_1^{\mathbf{w}} \wedge \cdots \wedge df_n^{\mathbf{w}} \neq 0$. Thus, (b) is equivalent to $\eta \neq 0$. Since $\eta \neq 0$ implies that $f_1^{\mathbf{w}}, \dots, f_n^{\mathbf{w}}$ are algebraically

independent over k , we see that (b) implies (c). By Corollary 2.3 (iii), (c) implies

$$k[f_1^{\mathbf{w}}, \dots, f_n^{\mathbf{w}}] = k[f_1, \dots, f_n]^{\mathbf{w}} = k[\mathbf{x}]^{\mathbf{w}} = k[\mathbf{x}],$$

and hence implies (d). Since $\eta = (\det JF^{\mathbf{w}})dx_1 \wedge \dots \wedge dx_n$, (d) implies $\eta \neq 0$, and hence implies (b). Therefore, (b), (c) and (d) are equivalent.

(iii) Set $g_i = F^{-1}(x_i)$ for $i = 1, \dots, n$. Then, we have $F(g_i) = x_i$, and hence $\deg_{\mathbf{w}} F(g_i) = w_i$. Since $\text{mdeg}_{\mathbf{w}} F = \mathbf{w}$ by assumption, $F^{\mathbf{w}}$ is injective by (ii). Thus, we get $\deg_{\mathbf{w}_F} g_i = \deg_{\mathbf{w}} F(g_i) = w_i$ by Corollary 2.3 (i). Since $\mathbf{w}_F = \text{mdeg}_{\mathbf{w}} F = \mathbf{w}$, it follows that $\deg_{\mathbf{w}} g_i = \deg_{\mathbf{w}_F} g_i = w_i$, proving $\text{mdeg}_{\mathbf{w}} F^{-1} = \mathbf{w}$. \square

Theorem 3.4. *Let F be an element of $\text{Aut}_k k[\mathbf{x}]$. If $\deg_{\mathbf{w}} F = |\mathbf{w}|$ holds for some $\mathbf{w} \in (\Gamma_+)^n$, then F belongs to $T_n(k)$.*

Proof. Without loss of generality, we may assume that F and \mathbf{w} satisfy (3.1). Since $\deg_{\mathbf{w}} F = |\mathbf{w}|$ by assumption, we have $\text{mdeg}_{\mathbf{w}} F = \mathbf{w}_{\sigma}$ for some $\sigma \in \mathfrak{S}_n$ by Theorem 3.3 (ii). Because of (3.1), this implies that $\deg_{\mathbf{w}} f_i = w_i$ for $i = 1, \dots, n$. We prove the assertion by induction on $r := \#\{w_1, \dots, w_n\}$. When $r = 1$, we have $\mathbf{w} = (w, \dots, w)$ for some $w \in \Gamma_+$. Since $w \deg f_i = \deg_{\mathbf{w}} f_i = w_i = w$, we know that $\deg f_i = 1$ for each i . Thus, F is an affine automorphism. Therefore, F belongs to $T_n(k)$. Assume that $r \geq 2$. Then, there exists $1 < l \leq n$ such that $w_{l-1} < w_l = \dots = w_n$. Since $\deg_{\mathbf{w}} f_{l-1} = w_{l-1} < w_l$, we know by Lemma 3.1 (ii) that $F_0 := (f_1, \dots, f_{l-1})$ is an automorphism of $k[x_1, \dots, x_{l-1}]$. Set $\mathbf{v} = (w_1, \dots, w_{l-1})$. Then, we have $\deg_{\mathbf{v}} f_i = \deg_{\mathbf{w}} f_i = w_i$ for $i = 1, \dots, l-1$, and so $\deg_{\mathbf{v}} F_0 = |\mathbf{v}|$. Hence, F_0 belongs to $T_{l-1}(k)$ by induction assumption. For $i = l, \dots, n$, we have

$$w_{l-1} < \deg_{\mathbf{w}} f_i = w_l = \dots = w_n.$$

Hence, we may write $f_i = \sum_{j=l}^n a_{i,j}x_j + g_i$ by Lemma 3.1 (iii), where $a_{i,j} \in k$ for each j , and $g_i \in k[x_1, \dots, x_{l-1}]$. Define $H \in T_n(k)$ by

$$H = (F_0^{-1}, x_l - F_0^{-1}(g_l), \dots, x_n - F_0^{-1}(g_n)).$$

Then, $F \circ H = (x_1, \dots, x_{l-1}, f_l - g_l, \dots, f_n - g_n)$ is an affine automorphism. Therefore, F belongs to $T_n(k)$. \square

Clearly, F does not necessary belong to $T_n(k)$ even if $\deg_{\mathbf{w}} F = |\mathbf{w}|$ for some $\mathbf{w} \in \Gamma^n \setminus (\Gamma_+)^n$, since $\deg_{\mathbf{w}} F = |\mathbf{w}|$ holds for any F for $\mathbf{w} = (0, \dots, 0)$.

4 Proof of Theorem 1.1

In this and the next section, we prove Theorem 1.1. The following theorem is due to the author.

Theorem 4.1 ([15, Theorem 1.4]). *Let $m \geq n$ and $f_1, \dots, f_m \in k[x_1, \dots, x_m]$ be such that $k[f_1, \dots, f_m] = k[x_1, \dots, x_m]$ and $k[\mathbf{x}] \not\subset k[f_2, \dots, f_m]$, and $S \subset k[\mathbf{x}] \setminus \{0\}$ such that $\text{trans.deg}_k k[S] = n$. Then, for each $\mathbf{w} \in \Gamma^n$, there exists $g \in S$ such that g does not divide $f^{\mathbf{w}}$ for any $f \in k[f_2, \dots, f_m] \cap k[\mathbf{x}] \setminus \{0\}$.*

Clearly, the conclusion of Theorem 4.1 holds for $S = \{x_1, \dots, x_n\}$. We mention that the case $m = n$ of Theorem 4.1 is implicit in [3]. When $m = n$, Theorem 4.1 implies that, for each coordinate f of $k[\mathbf{x}]$ over k and $\mathbf{w} \in \Gamma^n$, there exists $1 \leq i \leq n$ such that x_i does not divide $f^{\mathbf{w}}$.

The following lemma seems to be well known to the experts, but we give a proof in the next section for lack of a suitable reference.

Lemma 4.2. *For any $f_1, \dots, f_r \in k[\mathbf{x}] \setminus \{0\}$ with $r \geq 1$, and any totally ordered additive group $\Gamma \neq \{0\}$, the following assertions hold:*

- (i) *There exists $\mathbf{w} \in \Gamma^n$ such that $f_i^{\mathbf{w}}$ is a monomial for $i = 1, \dots, n$.*
- (ii) *For any $\mathbf{w}_1, \dots, \mathbf{w}_s \in \Gamma^n$ with $s \geq 1$, there exists $\mathbf{w} \in \Gamma^n$ such that*

$$(\dots (f_i^{\mathbf{w}_1})^{\mathbf{w}_2} \dots)^{\mathbf{w}_s} = f_i^{\mathbf{w}}$$

for $i = 1, \dots, r$. If \mathbf{w}_1 belongs to $(\Gamma_+)^n$, then we can take \mathbf{w} from $(\Gamma_+)^n$.

Now, we prove Theorem 1.1. Without loss of generality, we may assume that $\Gamma \neq \{0\}$. First, we show (ii). Set $f_0 = f$. By Lemma 4.2 (i) and (ii), there exist $\mathbf{v}', \mathbf{w}' \in \Gamma^n$ such that $((f_j^{\mathbf{w}})^{\mathbf{v}})^{\mathbf{v}'}$ is a monomial and is equal to $f_j^{\mathbf{w}'}$ for each $j \in J \cup \{0\}$. We show that there exists $i_0 \in I$ for which x_{i_0} does not divide $f_j^{\mathbf{w}'}$ for any $j \in J \cup \{0\}$. Then, it follows that x_{i_0} does not divide $(f_0^{\mathbf{w}})^{\mathbf{v}}$. Moreover, $\deg_{\mathbf{w}} f_j = \deg_{\mathbf{w}} ((f_j^{\mathbf{w}})^{\mathbf{v}})^{\mathbf{v}'}$ belongs to $\sum_{i \in I \setminus \{i_0\}} \mathbf{N}_0 w_i$ for each $j \in J$. Hence, i_0 belongs to I_0 . Thus, the proof of (ii) is completed.

Set $A_l = k[\{f_j \mid j \neq l\}]$ for each $l \in J^c := \{1, \dots, n\} \setminus J$. Since $\#I > \#J$ by assumption, $k[\mathbf{x}_I] := k[\{x_i \mid i \in I\}]$ is not contained in $k[\{f_j \mid j \in J\}] = \bigcap_{l \in J^c} A_l$. Hence, $k[\mathbf{x}_I]$ is not contained in A_{j_0} for some $j_0 \in J^c$. By Theorem 4.1, there exists $i_0 \in I$ such that x_{i_0} does not divide $f^{\mathbf{w}'}$ for any $f \in A_{j_0} \cap k[\mathbf{x}_I] \setminus \{0\}$. Since $k[\{f_j \mid j \in J\}]$ is contained in A_{j_0} by the choice of j_0 , and in $k[\mathbf{x}_I]$ by the definition of J , we have $k[\{f_j \mid j \in J\}] \subset A_{j_0} \cap k[\mathbf{x}_I]$. Thus, f_j belongs to $A_{j_0} \cap k[\mathbf{x}_I]$ for each $j \in J \cup \{0\}$. Therefore, x_{i_0} does not divide $f_j^{\mathbf{w}'}$ for any $j \in J \cup \{0\}$.

Next, we show (i). First, we prove (b) when $\#I > \#J$. Since $\prod_{j \in J} f_j$ is an element of $k[\{f_j \mid j \in J\}] \setminus \{0\}$, there exists $i \in I_0$ such that x_i does

not divide $((\prod_{j \in J} f_j)^{\mathbf{w}})^{\mathbf{v}} = \prod_{j \in J} (f_j^{\mathbf{w}})^{\mathbf{v}}$ by (ii). Then, x_i does not divide $(f_j^{\mathbf{w}})^{\mathbf{v}}$ for each $j \in J$, proving (b). It remains only to consider the case where $\#I = \#J$. Since $k[\mathbf{x}_I] = k[\{f_j \mid j \in J\}]$, we may assume that $I = J = \{1, \dots, n\}$. Thanks to Theorem 3.3 (i), it suffices to show that (b) holds when $\deg_{\mathbf{w}} F > |\mathbf{w}|$. By Lemma 4.2 (i) and (ii), there exist $\mathbf{v}', \mathbf{w}' \in \Gamma^n$ such that $((f_j^{\mathbf{w}})^{\mathbf{v}})^{\mathbf{v}'}$ is a monomial and is equal to $f_j^{\mathbf{w}'}$ for each $j \in J$. We show that there exists $1 \leq i_0 \leq n$ for which x_{i_0} does not divide $f_j^{\mathbf{w}'}$ for each $j \in J$. Then, it follows that i_0 belongs to I_0 , and x_{i_0} does not divide $(f_j^{\mathbf{w}})^{\mathbf{v}}$ for each $j \in J$ as in the proof of (ii). Thus, the proof is completed.

Suppose the contrary. Then, $f_1^{\mathbf{w}'} \cdots f_n^{\mathbf{w}'}$ is divisible by x_1, \dots, x_n . We claim that there exists $\sigma \in \mathfrak{S}_n$ for which $f_j^{\mathbf{w}'} = \alpha_j x_{\sigma(j)}^{u_j}$ for $j = 1, \dots, n$, where $\alpha_j \in k \setminus \{0\}$ and $u_j \geq 1$. In fact, if not, there exists j_0 such that $f_{j_0}^{\mathbf{w}'}$ belongs to $k \setminus \{0\}$, or $f_{j_0}^{\mathbf{w}'}$ is divisible by x_{i_1} and x_{i_2} for some $i_1 \neq i_2$. In either case, there exists l such that $(\prod_{j \neq l} f_j)^{\mathbf{w}'} = \prod_{j \neq l} f_j^{\mathbf{w}'}$ is divisible by x_1, \dots, x_n , contradicting Theorem 4.1 when $m = n$. Since $\deg_{\mathbf{w}} F > |\mathbf{w}|$ by assumption, $f_j^{\mathbf{w}}$'s are algebraically dependent over k by Theorem 3.3 (ii). By Corollary 2.3 (ii), it follows that $(f_j^{\mathbf{w}})^{\mathbf{v}}$'s are algebraically dependent over k , and hence so are $((f_j^{\mathbf{w}})^{\mathbf{v}})^{\mathbf{v}'}$'s. This contradicts that $((f_j^{\mathbf{w}})^{\mathbf{v}})^{\mathbf{v}'} = f_j^{\mathbf{w}'} = \alpha_j x_{\sigma(j)}^{u_j}$ for each j .

5 Approximation of a weight

The goal of this section is to prove Lemma 4.2. We define $a \cdot \mathbf{w} = a_1 w_1 + \cdots + a_n w_n \in \Gamma$ for each $a = (a_1, \dots, a_n) \in \mathbf{Z}^n$ and $\mathbf{w} \in \Gamma^n$.

Lemma 5.1. *Let S be a finite subset of \mathbf{Z}^n for which there exists $\mathbf{w} \in \Gamma^n$ such that $a \cdot \mathbf{w} > 0$ for each $a \in S$. Then, there exists $\mathbf{v} \in \mathbf{Z}^n$ such that $a \cdot \mathbf{v} > 0$ for each $a \in S$.*

Proof. Let C be the set of $\mathbf{v} \in \mathbf{R}^n$ such that $a \cdot \mathbf{v} > 0$ for each $a \in S$. We show that $C \neq \emptyset$. Then, it follows that $C \cap \mathbf{Q}^n \neq \emptyset$, since C is an open subset of \mathbf{R}^n for the Euclidean topology. Since C is a cone, this implies that $C \cap \mathbf{Z}^n \neq \emptyset$. Thus, the proof is completed. We define $P = \{\sum_{a \in S} \lambda_a a \mid (\lambda_a)_a \in (\mathbf{R}_{\geq 0})^S\}$, where $\mathbf{R}_{\geq 0} := \{\lambda \in \mathbf{R} \mid \lambda \geq 0\}$. Then, $F := P \cap \{-a \mid a \in P\}$ is a *face* of P , i.e., there exists $\mathbf{v} \in \mathbf{R}^n$ such that $a \cdot \mathbf{v} = 0$ and $b \cdot \mathbf{v} > 0$ for each $a \in F$ and $b \in P \setminus F$ (cf. [22, Proposition A5]). We show that \mathbf{v} belongs to C . By the choice of \mathbf{v} , it suffices to check that S is contained in $P \setminus F$. Suppose the contrary. Then, we have $S \cap F \neq \emptyset$, since S is contained in P . Hence, there exist $a \in S$ and $(\lambda_b)_b \in (\mathbf{R}_{\geq 0})^S$ such that $a = -\sum_{b \in S} \lambda_b b$. Since S is a subset of \mathbf{Z}^n , we may take $(\lambda_b)_b$ from $(\mathbf{Q} \cap \mathbf{R}_{\geq 0})^S$. Choose $l \in \mathbf{N}$ so that $(l\lambda_b)_b$ belongs to $(\mathbf{N}_0)^S$. Then, we have $0 < l(a \cdot \mathbf{w}) = -\sum_{b \in S} l\lambda_b (b \cdot \mathbf{w}) \leq 0$

by the assumption that $b \cdot \mathbf{w} > 0$ for each $b \in S$. This is a contradiction. Therefore, \mathbf{v} belongs to C . \square

Let Γ and Γ' be totally ordered additive groups. For $\mathbf{w} \in \Gamma^n$, $\mathbf{w}' \in (\Gamma')^n$ and $S \subset \mathbf{Z}^n$, we define $\mathbf{w} \sim_S \mathbf{w}'$ if, for each $a, b \in S$, we have $a \cdot \mathbf{w} \geq b \cdot \mathbf{w}$ if and only if $a \cdot \mathbf{w}' \geq b \cdot \mathbf{w}'$. For

$$f = \sum_{i_1, \dots, i_n} \alpha_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n} \in k[\mathbf{x}]$$

with $\alpha_{i_1, \dots, i_n} \in k$, we define $\text{supp } f$ to be the set of $(i_1, \dots, i_n) \in (\mathbf{N}_0)^n$ such that $\alpha_{i_1, \dots, i_n} \neq 0$. Then, we have $f^{\mathbf{w}} = f^{\mathbf{w}'}$ if $\mathbf{w} \sim_S \mathbf{w}'$ for $S = \text{supp } f$. More generally, set $S = \bigcup_{i=1}^r \text{supp } f_i$ for $f_1, \dots, f_r \in k[\mathbf{x}]$ with $r \geq 1$. Then, we have $f_i^{\mathbf{w}} = f_i^{\mathbf{w}'}$ for $i = 1, \dots, r$ if $\mathbf{w} \sim_S \mathbf{w}'$.

Proposition 5.2 (Approximation of a weight). *For any finite subset S of \mathbf{Z}^n and $\mathbf{w} \in \Gamma^n$, there exists $\mathbf{v} \in \mathbf{Z}^n$ such that $\mathbf{w} \sim_S \mathbf{v}$.*

Proof. Let T_0 (resp. T_1) be the set of $a - b$ for $a, b \in S$ such that $a \cdot \mathbf{w} = b \cdot \mathbf{w}$ (resp. $a \cdot \mathbf{w} > b \cdot \mathbf{w}$). It suffices to construct $\mathbf{v} \in \mathbf{Z}^n$ such that $a \cdot \mathbf{v} = 0$ and $b \cdot \mathbf{v} > 0$ for each $a \in T_0$ and $b \in T_1$. Since Γ is torsion-free, the \mathbf{Z} -submodule Γ' of Γ generated by w_1, \dots, w_n is a free \mathbf{Z} -module of finite rank. Take a \mathbf{Z} -basis u_1, \dots, u_r of Γ' , and put $\mathbf{u} = (u_1, \dots, u_r)$. Then, we may write $\mathbf{w} = \mathbf{u}U$, where U is an $r \times n$ matrix with integer entries. Let U' be the transposition of U . Then, we have $(aU') \cdot \mathbf{u} = a \cdot (\mathbf{u}U) = a \cdot \mathbf{w} = 0$ for each $a \in T_0$. Since u_1, \dots, u_r are linearly independent over \mathbf{Z} , it follows that $aU' = 0$ for each $a \in T_0$. Since $(aU') \cdot \mathbf{u} = a \cdot \mathbf{w} > 0$ for each $a \in T_1$, and $\{aU' \mid a \in T_1\}$ is a finite subset of \mathbf{Z}^r , there exists $\mathbf{v}' \in \mathbf{Z}^r$ such that $(aU') \cdot \mathbf{v}' > 0$ for each $a \in T_1$ by Lemma 5.1. Then, $\mathbf{v} := \mathbf{v}'U$ is an element of \mathbf{Z}^n such that $a \cdot \mathbf{v} = (aU') \cdot \mathbf{v}' = 0$ and $b \cdot \mathbf{v} = (bU') \cdot \mathbf{v}' > 0$ for each $a \in T_0$ and $b \in T_1$. Therefore, \mathbf{v} satisfies the required condition. \square

Under the assumption of Proposition 5.2, there exists an element $\mathbf{v} = (v_1, \dots, v_n)$ of \mathbf{Z}^n such that $\mathbf{w} \sim_S \mathbf{v}$ and $v_i > 0$ (resp. $v_i < 0$) if and only if $w_i > 0$ (resp. $w_i < 0$) for $i = 1, \dots, n$ for the following reason. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the coordinate unit vectors of \mathbf{R}^n , and let $S' = S \cup \{0, \mathbf{e}_1, \dots, \mathbf{e}_n\}$. By Proposition 5.2, there exists $\mathbf{v} \in \mathbf{Z}^n$ such that $\mathbf{w} \sim_{S'} \mathbf{v}$. Then, this \mathbf{v} has the property stated above, since $\mathbf{e}_i \cdot \mathbf{v} = v_i$ and $\mathbf{e}_i \cdot \mathbf{w} = w_i$ for each i . In particular, if \mathbf{w} is an element of $(\Gamma_+)^n$, then we can take \mathbf{v} from \mathbf{N}^n .

Now, let us prove Lemma 4.2. To show (i), take any $\mathbf{u} \in \mathbf{R}^n$ whose components are linearly independent over \mathbf{Q} . Then, $f_i^{\mathbf{u}}$ is a monomial for each i . Set $S = \bigcup_{i=1}^r \text{supp } f_i$. Then, there exists $\mathbf{v} = (v_1, \dots, v_n) \in \mathbf{Z}^n$ such that $\mathbf{v} \sim_S \mathbf{u}$ by Proposition 5.2. Since $\Gamma \neq \{0\}$ by assumption, we may find

$w \in \Gamma_+$. Then, $\mathbf{w} := (v_1 w, \dots, v_n w)$ is an element of Γ^n such that $\mathbf{w} \sim_{\mathbf{Z}^n} \mathbf{v}$. Since $\mathbf{v} \sim_S \mathbf{u}$, we get $\mathbf{w} \sim_S \mathbf{u}$. Therefore, $f_i^{\mathbf{w}} = f_i^{\mathbf{u}}$ is a monomial for each i , proving (i).

Next, we prove (ii) by induction on s . When $s = 1$, the assertion is clear. Assume that $s \geq 2$. Then, by induction assumption, there exists $\mathbf{w}' \in \Gamma^n$ such that $(\dots (f_i^{\mathbf{w}_1})^{\mathbf{w}_2} \dots)^{\mathbf{w}_{s-1}} = f_i^{\mathbf{w}'}$ for $i = 1, \dots, r$. By Proposition 5.2, there exist $\mathbf{v}', \mathbf{v}'' \in \mathbf{Z}^n$ such that $\mathbf{w}' \sim_S \mathbf{v}'$ and $\mathbf{w}_s \sim_S \mathbf{v}''$. Then, we have

$$h_i := ((\dots (f_i^{\mathbf{w}_1})^{\mathbf{w}_2} \dots)^{\mathbf{w}_{s-1}})^{\mathbf{w}_s} = (f_i^{\mathbf{w}'})^{\mathbf{w}_s} = (f_i^{\mathbf{v}'})^{\mathbf{v}''}$$

for $i = 1, \dots, r$. We define $\mathbf{v}(t) = \mathbf{v}' + t\mathbf{v}'' \in \mathbf{R}^n$ for each $t \in \mathbf{R}$. Then, we have

$$(a - b) \cdot \mathbf{v}(t) = (a - b) \cdot \mathbf{v}' + (a - b) \cdot (t\mathbf{v}'') = t((a - b) \cdot \mathbf{v}'')$$

for each $a, b \in T_i := \text{supp } f_i^{\mathbf{v}'}$, since $a \cdot \mathbf{v}' = b \cdot \mathbf{v}' = \deg_{\mathbf{v}'} f_i$. Hence, if $t > 0$, then we have $\mathbf{v}(t) \sim_{T_i} \mathbf{v}''$, and so $(f_i^{\mathbf{v}'})^{\mathbf{v}(t)} = (f_i^{\mathbf{v}'})^{\mathbf{v}''}$. Since $\deg_{\mathbf{v}(t)} f_i^{\mathbf{v}'}$ and $\deg_{\mathbf{v}(t)} (f_i - f_i^{\mathbf{v}'})$ are continuous functions in t satisfying

$$\deg_{\mathbf{v}(0)} f_i^{\mathbf{v}'} = \deg_{\mathbf{v}'} f_i^{\mathbf{v}'} > \deg_{\mathbf{v}'} (f_i - f_i^{\mathbf{v}'}) = \deg_{\mathbf{v}(0)} (f_i - f_i^{\mathbf{v}'}),$$

there exists $t_0 > 0$ such that $\deg_{\mathbf{v}(t)} f_i^{\mathbf{v}'} > \deg_{\mathbf{v}(t)} (f_i - f_i^{\mathbf{v}'})$ for $i = 1, \dots, r$ for any $0 < t < t_0$. Here, we regard $\deg_{\mathbf{v}(t)} (f_i - f_i^{\mathbf{v}'})$ as a constant function with value $-\infty$ if $f_i^{\mathbf{v}'} = f_i$. Then, for any $0 < t < t_0$, we have

$$f_i^{\mathbf{v}(t)} = (f_i^{\mathbf{v}'} + (f_i - f_i^{\mathbf{v}'}))^{\mathbf{v}(t)} = (f_i^{\mathbf{v}'})^{\mathbf{v}(t)} = (f_i^{\mathbf{v}'})^{\mathbf{v}''} = h_i$$

for $i = 1, \dots, r$. Now, take any $w \in \Gamma_+$ and $t \in \mathbf{Q}$ with $0 < t < t_0$. Let $u \in \mathbf{N}$ be such that $(u_1, \dots, u_n) := u\mathbf{v}(t)$ belongs to \mathbf{Z}^n . Then, $\mathbf{w} := (u_1 w, \dots, u_n w)$ is an element of Γ^n such that $\mathbf{w} \sim_{\mathbf{Z}^n} \mathbf{v}(t)$, and hence $f_i^{\mathbf{w}} = f_i^{\mathbf{v}(t)} = h_i$ for $i = 1, \dots, r$.

If \mathbf{w}_1 is an element of $(\Gamma_+)^n$, then we can take \mathbf{w}' from $(\Gamma_+)^n$ by induction assumption. Then, \mathbf{v}' can be taken from \mathbf{N}^n as mentioned after Proposition 5.2. In this case, all the components of $\mathbf{v}(t)$ become positive for sufficiently small $t > 0$. For such t , the element \mathbf{w} of Γ^n constructed above belongs to $(\Gamma_+)^n$. This completes the proof of Lemma 4.2.

6 Van der Kulk's theorem

Assume that $n = 2$ and k is a field. Then, $\deg f_1 \mid \deg f_2$ or $\deg f_2 \mid \deg f_1$ holds for each $F \in \text{Aut}_k k[\mathbf{x}]$ by van der Kulk [10]. If $d_i := \deg_{x_i} f > 0$ for

$i = 1, 2$ for a coordinate f of $k[\mathbf{x}]$ over k , then the following statements hold by Makar-Limanov [17] (see also Dicks [4]):

- (i) $d_1 \mid d_2$ or $d_2 \mid d_1$.
- (ii) $(d_1, 0)$ and $(0, d_2)$ belong to $\text{supp } f$.
- (iii) $\text{supp } f$ is contained in the convex hull of $(0, 0)$, $(d_1, 0)$ and $(0, d_2)$ in \mathbf{R}^2 .

In this section, we revisit the well-known results stated above. For each $f_1, f_2 \in k[\mathbf{x}]$, we denote $f_1 \approx f_2$ if f_1 and f_2 are linearly dependent over k . Clearly, $f_1 \approx f_2$ implies $\deg_{\mathbf{w}} f_1 = \deg_{\mathbf{w}} f_2$ for any $\mathbf{w} \in \Gamma^n$.

The following lemma is a weighted version of van der Kulk's theorem, which is proved by using Makar-Limanov's theorem.

Lemma 6.1. *Assume that $n = 2$ and k is a field. Let $F \in \text{Aut}_k k[\mathbf{x}]$ and $\mathbf{w} \in (\Gamma_{\geq 0})^2$ be such that $\deg_{\mathbf{w}} F > |\mathbf{w}|$. Then, $\deg_{\mathbf{w}} f_1$ and $\deg_{\mathbf{w}} f_2$ are positive, and $f_1^{\mathbf{w}} \approx (f_2^{\mathbf{w}})^u$ or $f_2^{\mathbf{w}} \approx (f_1^{\mathbf{w}})^u$ holds for some $u \geq 1$.*

Proof. Since $\deg_{\mathbf{w}} F > |\mathbf{w}|$, we have $w_1 > 0$ and $w_2 \geq 0$, or $w_1 \geq 0$ and $w_2 > 0$. First, assume that f_1 and f_2 do not belong to $k[x_1]$ or $k[x_2]$. Then, $\deg_{\mathbf{w}} f_1$ and $\deg_{\mathbf{w}} f_2$ are positive. Since $\deg_{\mathbf{w}} F > |\mathbf{w}|$, we know by Theorem 3.3 (ii) that $f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ are algebraically dependent over k . Hence, $\deg_{\mathbf{w}} f_1$ and $\deg_{\mathbf{w}} f_2$ are linearly dependent over \mathbf{Z} by Corollary 2.4. Since $\deg_{\mathbf{w}} f_i > 0$ for $i = 1, 2$, there exist $u_1, u_2 \in \mathbf{N}$ such that $\gcd(u_1, u_2) = 1$ and $u_1 \deg_{\mathbf{w}} f_1 = u_2 \deg_{\mathbf{w}} f_2$. We show that $(f_1^{\mathbf{w}})^{u_1} \approx (f_2^{\mathbf{w}})^{u_2}$. Observe that a Γ -grading $k[\mathbf{x}][1/f_2^{\mathbf{w}}] = \bigoplus_{\gamma \in \Gamma} k[\mathbf{x}][1/f_2^{\mathbf{w}}]_{\gamma}$ is induced from the \mathbf{w} -weighted Γ -grading of $k[\mathbf{x}]$. Since $h := (f_1^{\mathbf{w}})^{u_1} / (f_2^{\mathbf{w}})^{u_2}$ belongs to $k[\mathbf{x}][1/f_2^{\mathbf{w}}]_0$, and $\deg_{\mathbf{w}} f_2^{\mathbf{w}} > 0$, we see that $k[h][f_2^{\mathbf{w}}]$ is the polynomial ring in $f_2^{\mathbf{w}}$ over $k[h]$. Since h and $f_2^{\mathbf{w}}$ are algebraically dependent over k , it follows that h belongs to k . Therefore, we get $(f_1^{\mathbf{w}})^{u_1} = h(f_2^{\mathbf{w}})^{u_2} \approx (f_2^{\mathbf{w}})^{u_2}$. It remains only to show that $u_1 = 1$ or $u_2 = 1$. Set $g_i = F^{-1}(x_i)$ for $i = 1, 2$. Then, we have

$$\deg_{\mathbf{w}_F} g_1 + \deg_{\mathbf{w}_F} g_2 = \deg_{\mathbf{w}_F} F^{-1} \geq |\mathbf{w}_F| = \deg_{\mathbf{w}} F > |\mathbf{w}| = w_1 + w_2$$

by Theorem 3.3 (i). Hence, $\deg_{\mathbf{w}_F} g_l > w_l = \deg_{\mathbf{w}} x_l = \deg_{\mathbf{w}} F(g_l)$ holds for some $l \in \{1, 2\}$. Then, we have $F^{\mathbf{w}}(g_l^{\mathbf{w}_F}) = 0$ by Proposition 2.2 (ii). This implies that $g_l^{\mathbf{w}_F}$ is not a monomial. Note that g_l does not belong to $k[x_1]$ or $k[x_2]$, for otherwise f_1 or f_2 belongs to $k[x_l]$, a contradiction. Hence, $d_i := \deg_{x_i} g_l > 0$ holds for $i = 1, 2$. Thus, the statements (i), (ii) and (iii) above hold for $f = g_l$. Since \mathbf{w}_F belongs to $(\Gamma_+)^2$, and $g_l^{\mathbf{w}_F}$ is not a monomial, we see from (ii) and (iii) that $(d_1, 0) \cdot \mathbf{w}_F$ and $(0, d_2) \cdot \mathbf{w}_F$ are both equal to $\deg_{\mathbf{w}_F} g_l$. Hence, we have $d_1 \deg_{\mathbf{w}} f_1 = d_2 \deg_{\mathbf{w}} f_2$. Since $u_1 \deg_{\mathbf{w}} f_1 = u_2 \deg_{\mathbf{w}} f_2$ and $\gcd(u_1, u_2) = 1$, we conclude from (i) that $u_1 = 1$ or $u_2 = 1$.

Next, assume that f_{i_1} belongs to $k[x_{j_1}]$ for some $i_1, j_1 \in \{1, 2\}$. Then, we may write $f_{i_1} = \alpha_1 x_{j_1} + \beta$ and $f_{i_2} = \alpha_2 x_{j_2} + p$. Here, $\alpha_1, \alpha_2 \in k^{\times}$, $\beta \in k$

and $p \in k[x_{j_1}]$, and $i_2, j_2 \in \{1, 2\}$ are such that $i_2 \neq i_1$ and $j_2 \neq j_1$. If $w_{j_1} = 0$, then $\deg_{\mathbf{w}} F = \deg_{\mathbf{w}} f_{i_2} = w_{j_2} = |\mathbf{w}|$, a contradiction. Hence, we have $w_{j_1} > 0$, and so $f_{i_1}^{\mathbf{w}} = \alpha_1 x_{j_1}$. Since $\deg_{\mathbf{w}} f_{i_1} = w_{j_1}$ and $\deg_{\mathbf{w}} F > |\mathbf{w}|$, we know that $\deg_{\mathbf{w}} f_{i_2} > w_{j_2}$. This implies that $f_{i_2}^{\mathbf{w}} = p^{\mathbf{w}} \approx x_{j_1}^u$ for some $u \geq 1$. Since $f_{i_1}^{\mathbf{w}} \approx x_{j_1}$, it follows that $f_{i_2}^{\mathbf{w}} \approx (f_{i_1}^{\mathbf{w}})^u$. \square

We mention that the author [12, Corollary 4.4] proved a statement similar to Lemma 6.1 as an application of the generalized Shestakov-Umirbaev inequality when $\Gamma = \mathbf{Z}$ and k is a field of characteristic zero.

Now, assume that $n \geq 2$ and k is a domain. Let us consider the following conditions for $F \in \text{Aut}_k k[\mathbf{x}]$ and $\mathbf{w} \in (\Gamma_{\geq 0})^n$:

- (a) $f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ belong to $k[x_1, x_2]$.
- (b) $\deg_{\mathbf{w}} f_1 + \deg_{\mathbf{w}} f_2 > w_1 + w_2$.
- (c) $\deg_{\mathbf{w}} f_i = w_i$ for $i = 3, \dots, n$.
- (d) $k[x_1, x_2, f_3, \dots, f_n] = k[\mathbf{x}]$.

Then, we have the following theorem.

Theorem 6.2. *Assume that $n \geq 2$ and k is a domain. If $F \in \text{Aut}_k k[\mathbf{x}]$ and $\mathbf{w} \in (\Gamma_{\geq 0})^n$ satisfy (a) through (d), then the following assertions hold:*

- (i) $f_1^{\mathbf{w}} \approx (f_2^{\mathbf{w}})^u$ or $f_2^{\mathbf{w}} \approx (f_1^{\mathbf{w}})^u$ for some $u \geq 1$.
- (ii) $\deg_{\mathbf{w}} f_l > 0$ for $l = 1, 2$.
- (iii) *For any commutative ring κ , there exists $G \in E_n^{\mathbf{w}}(\kappa)$ such that $g_i = x_i$ for $i = 3, \dots, n$ and $G \sim_{\mathbf{w}} F$. In particular, $\text{mdeg}_{\mathbf{w}} F$ belongs to $|E_n^{\mathbf{w}}|$.*

Proof. By replacing k with the field of fractions of k , we may assume that k is a field. We may also assume that $f_i = x_i$ for each $i \geq 3$ for the following reason. By (d), we can define an element of $\text{Aut}_k k[\mathbf{x}]$ by $(x_1, x_2, f_3, \dots, f_n)$, whose multidegree is equal to \mathbf{w} by (c). The inverse of this automorphism has the form $H = (x_1, x_2, h_3, \dots, h_n)$ for some $h_3, \dots, h_n \in k[\mathbf{x}]$, and satisfies

$$H \circ F = (H(f_1), H(f_2), x_3, \dots, x_n).$$

By Theorem 3.3 (iii), we have $\mathbf{w}_H = \text{mdeg}_{\mathbf{w}} H = \mathbf{w}$. Hence, we know by Theorem 3.3 (ii) and Corollary 2.3 (i) that $H(f)^{\mathbf{w}} = H^{\mathbf{w}}(f^{\mathbf{w}_H}) = H^{\mathbf{w}}(f^{\mathbf{w}})$ for each $f \in k[\mathbf{x}]$. Since $H^{\mathbf{w}} = (x_1, x_2, h_3^{\mathbf{w}}, \dots, h_n^{\mathbf{w}})$ fixes x_1 and x_2 , we have $H^{\mathbf{w}}(f_i^{\mathbf{w}}) = f_i^{\mathbf{w}}$ for $i = 1, 2$ by (a). Thus, $H(f_i)^{\mathbf{w}} = f_i^{\mathbf{w}}$ holds for $i = 1, 2$. Therefore, by replacing F with $H \circ F$, we may assume that $f_i = x_i$ for each $i \geq 3$.

Set $\tilde{\mathbf{w}} = (w_1, w_2)$, $\mathbf{w}' = (w_1, w_2, 0, \dots, 0)$ and $K = k(x_3, \dots, x_n)$. Then, $\deg_{\tilde{\mathbf{w}}} f$ and $f^{\tilde{\mathbf{w}}}$ can be defined for each $f \in k[\mathbf{x}]$ as an element of $K[x_1, x_2]$. We note that $\deg_{\tilde{\mathbf{w}}} f = \deg_{\mathbf{w}'} f$ and $f^{\tilde{\mathbf{w}}} = f^{\mathbf{w}'}$ by definition. Since $w_i \geq 0$

for each i , we have $f_l^{\mathbf{w}} = f_l^{\mathbf{w}'}$ and $\deg_{\mathbf{w}} f_l = \deg_{\mathbf{w}'} f_l$ for $l = 1, 2$ in view of (a). Hence, $f_l^{\tilde{\mathbf{w}}} = f_l^{\mathbf{w}}$ and $\deg_{\tilde{\mathbf{w}}} f_l = \deg_{\mathbf{w}} f_l$ hold for $l = 1, 2$. Since $f_i = x_i$ for $i = 3, \dots, n$ by assumption, we can define $\tilde{F} \in \text{Aut}_K K[x_1, x_2]$ by $\tilde{F} = (f_1, f_2)$. Then, we have

$$\deg_{\tilde{\mathbf{w}}} \tilde{F} = \deg_{\tilde{\mathbf{w}}} f_1 + \deg_{\tilde{\mathbf{w}}} f_2 = \deg_{\mathbf{w}} f_1 + \deg_{\mathbf{w}} f_2 > w_1 + w_2 = |\tilde{\mathbf{w}}|$$

by (b). Thus, we obtain the following statements by Lemma 6.1:

- (i') $f_i^{\tilde{\mathbf{w}}} = c(f_j^{\tilde{\mathbf{w}}})^u$ for some $(i, j) \in \{(1, 2), (2, 1)\}$, $c \in K^\times$ and $u \geq 1$.
- (ii') $\deg_{\tilde{\mathbf{w}}} f_l > 0$ for $l = 1, 2$.

Since $f_l^{\tilde{\mathbf{w}}} = f_l^{\mathbf{w}}$ for $l = 1, 2$, we know by (i') that $f_i^{\mathbf{w}} = c(f_j^{\mathbf{w}})^u$. Hence, c belongs to $k(x_1, x_2)$ by (a), and thus to $k(x_1, x_2) \cap K^\times = k^\times$. Therefore, we get (i). Similarly, (ii) follows from (ii'). We show (iii). By Lemma 4.2 (i) and (ii), there exist $\mathbf{v}, \mathbf{w}' \in \Gamma^n$ such that $(f_j^{\mathbf{w}})^{\mathbf{v}}$ is a monomial and is equal to $f_j^{\mathbf{w}'}$. Because of (a), we may write $(f_j^{\mathbf{w}})^{\mathbf{v}} = \alpha x_1^{l_1} x_2^{l_2}$, where $\alpha \in k^\times$ and $l_1, l_2 \in \mathbf{N}_0$. Since $(l_1, l_2) \cdot \mathbf{w} = \deg_{\mathbf{w}} f_j > 0$ by (ii), we have $(l_1, l_2) \neq (0, 0)$. We claim that $l_1 = 0$ or $l_2 = 0$. In fact, if not, $(f_j x_3 \cdots x_n)^{\mathbf{w}'} = \alpha x_1^{l_1} x_2^{l_2} x_3 \cdots x_n$ is divisible by x_1, \dots, x_n , contradicting Theorem 4.1. Let $r, s \in \{1, 2\}$ be such that $l_r \geq 1$ and $l_s = 0$. Then, we have $\deg_{\mathbf{w}} f_j = l_r w_r$, and so

$$\deg_{\mathbf{w}} f_i = u \deg_{\mathbf{w}} f_j = u l_r w_r \geq w_r$$

by (i). First, assume that $\deg_{\mathbf{w}} f_j \geq w_s$. Then, we have $\deg_{\mathbf{w}} f_j = \deg_{\mathbf{w}}(x_s + x_r^{l_r})$ and $\deg_{\mathbf{w}} f_i = \deg_{\mathbf{w}}(x_s + x_r^{l_r})^u$. When $\deg_{\mathbf{w}} f_i > w_r$, we define $G \in E_n(\kappa)$ by

$$g_i = x_r + (x_s + x_r^{l_r})^u, \quad g_j = x_s + x_r^{l_r}$$

and $g_l = x_l$ for $l = 3, \dots, n$. Then, G belongs to $E_n^{\mathbf{w}}(\kappa)$ and satisfies $G \sim_{\mathbf{w}} F$. If $\deg_{\mathbf{w}} f_i = w_r$, then $G \sim_{\mathbf{w}} F$ holds for $G \in E_n^{\mathbf{w}}(\kappa)$ defined by $g_i = x_r$, $g_j = x_s + x_r^{l_r}$ and $g_l = x_l$ for $l = 3, \dots, n$. Next, assume that $\deg_{\mathbf{w}} f_j < w_s$. Then, f_j belongs to $K[x_r]$. Since f_j is a coordinate of $K[x_1, x_2]$ over K , this implies that $\deg_{x_r} f_j = 1$. Since $f_j^{\mathbf{w}}$ belongs to $K[x_r] \cap k[x_1, x_2] = k[x_r]$, we get $\deg_{\mathbf{w}} f_j = \deg_{\mathbf{w}} f_j^{\mathbf{w}} = w_r$, and so $\deg_{\mathbf{w}} f_i = u w_r$. In view of (b), we have $\deg_{\mathbf{w}} f_i > w_s$. Hence, $G \sim_{\mathbf{w}} F$ holds for $G \in E_n^{\mathbf{w}}(\kappa)$ defined by $g_i = x_s + x_r^u$, $g_j = x_r$ and $g_l = x_l$ for $l = 3, \dots, n$. \square

In the case of $n = 2$, the conditions (a), (c) and (d) are obvious. Hence, if $\deg_{\mathbf{w}} F > |\mathbf{w}|$ for $F \in \text{Aut}_k k[\mathbf{x}]$ and $\mathbf{w} \in (\Gamma_{\geq 0})^2$, then $\deg_{\mathbf{w}} F$ belongs to $|E_2^{\mathbf{w}}|$ by Theorem 6.2 (iii). The same holds when $\deg_{\mathbf{w}} F = |\mathbf{w}|$ as remarked before Theorem 3.3. Therefore, $\text{mdeg}_{\mathbf{w}}(\text{Aut}_k k[\mathbf{x}])$ is contained in $|E_2^{\mathbf{w}}|$. Since $|E_2^{\mathbf{w}}|$ is contained in the subset $\text{mdeg}_{\mathbf{w}} E_2(k)$ of $\text{mdeg}_{\mathbf{w}}(\text{Aut}_k k[\mathbf{x}])$, we get $\text{mdeg}_{\mathbf{w}}(\text{Aut}_k k[\mathbf{x}]) = |E_2^{\mathbf{w}}|$.

Corollary 6.3. *Assume that $n = 3$ and k is a domain. Then, the following assertions hold for each $F \in \text{Aut}_k k[\mathbf{x}]$:*

- (i) *If f_{i_1} and f_{i_2} belong to $k[x_{j_1}, x_{j_2}]$ for some $1 \leq i_1 < i_2 \leq 3$ and $1 \leq j_1 < j_2 \leq 3$, then $\text{mdeg}_{\mathbf{w}} F$ belongs to $|\mathbf{E}_3^{\mathbf{w}}|$ for any $\mathbf{w} \in (\Gamma_{\geq 0})^3$.*
- (ii) *Assume that f_j belongs to $k[x_i]$ for some $i, j \in \{1, 2, 3\}$. Then, for any commutative ring κ and $\mathbf{w} \in (\Gamma_{\geq 0})^3$ with $w_i = 0$, there exists $G \in \mathbf{E}_3^{\mathbf{w}}(\kappa)$ such that $g_j = x_i$ and $G \sim_{\mathbf{w}} F$.*

Proof. (i) We may assume that $(i_1, i_2) = (j_1, j_2) = (1, 2)$. Then, we have $k[f_1, f_2] = k[x_1, x_2]$ and $f_3 = \alpha x_3 + p$ for some $\alpha \in k^\times$ and $p \in k[x_1, x_2]$. Set $\mathbf{v} = (w_1, w_2)$ and take any commutative ring κ . Then, there exists $(g_1, g_2) \in \mathbf{E}_2^{\mathbf{v}}(\kappa)$ such that $(g_1, g_2) \sim_{\mathbf{v}} (f_1, f_2)$ by the discussion above. Define $q \in \kappa[x_1, x_2]$ by $q = 0$ if $p = 0$, and $q = x_1^{l_1} x_2^{l_2}$ if $p \neq 0$, where $l_1, l_2 \in \mathbf{N}_0$ are such that $\deg_{\mathbf{w}} p = l_1 w_1 + l_2 w_2$. Then, $G := (g_1, g_2, x_3 + q)$ is an element of $\mathbf{E}_3^{\mathbf{w}}(\kappa)$ such that $G \sim_{\mathbf{w}} F$. Therefore, $\text{mdeg}_{\mathbf{w}} F$ belongs to $|\mathbf{E}_3^{\mathbf{w}}|$.

(ii) We may assume that $i = j = 3$. Set $\mathbf{v} = (w_1, w_2)$. Then, $\deg_{\mathbf{w}} f_l$ is equal to the \mathbf{v} -degree of f_l as a polynomial in x_1 and x_2 over $k' := k[x_3]$ for each l . Since f_3 belongs to k' by assumption, we have $k'[f_1, f_2] = k'[x_1, x_2]$. Hence, there exists $(g_1, g_2) \in \mathbf{E}_2^{\mathbf{v}}(\kappa)$ such that $(g_1, g_2) \sim_{\mathbf{v}} (f_1, f_2)$ by the discussion above. Then, $G = (g_1, g_2, x_3)$ is an element of $\mathbf{E}_3^{\mathbf{w}}(\kappa)$ such that $G \sim_{\mathbf{w}} F$. \square

7 Proofs of Theorems 1.6 and 1.7.

The goal of this section is to prove Theorems 1.6 and 1.7. For this purpose, we use the following theorem which is implicit in Asanuma [1] (cf. [15, Section 3]).

Theorem 7.1. *If k is an integrally closed domain, then every stable coordinate of $k[x_1, x_2]$ over k is a coordinate of $k[x_1, x_2]$ over k .*

We mention that Shpilrain-Yu [20] showed Theorem 7.1 when k is a field of characteristic zero in a different manner.

We use the following proposition to prove Theorems 1.6, 1.7 and 1.10 (i).

Proposition 7.2. *Assume that $n = 3$ and k is a domain. Let $F \in \text{Aut}_k k[\mathbf{x}]$ be such that f_1 belongs to $k[x_1, x_2]$, and $f_3 = ax_3 + p$ for some $a \in k \setminus \{0\}$ and $p \in k[x_1, x_2]$.*

- (i) *If k is a field, then F belongs to $\mathbf{T}_3(k)$.*
- (ii) *Let $\mathbf{w} \in (\Gamma_{\geq 0})^3$ be such that $\deg_{\mathbf{w}} p \leq w_3$. Then, for any commutative ring κ , there exists $G \in \mathbf{E}_3^{\mathbf{w}}(\kappa)$ such that $g_3 = x_3$ and $G \sim_{\mathbf{w}} F$. In particular, $\text{mdeg}_{\mathbf{w}} F$ belongs to $|\mathbf{E}_3^{\mathbf{w}}|$.*

Proof. By replacing k with the field of fractions of k , we may assume that k is a field. Then, we can define $\psi \in T_3(k)$ by $\psi(x_i) = x_i$ for $i = 1, 2$ and $\psi(x_3) = f_3$. Since f_1 belongs to $k[x_1, x_2]$ by assumption, there exists $\phi \in \text{Aut}_k k[x_1, x_2]$ such that $\phi(x_1) = f_1$ by Theorem 7.1. By Jung [6] and van der Kulk [10], we have $\text{Aut}_k k[x_1, x_2] = T_2(k)$. Hence, we can extend ϕ to an element of $T_3(k)$ by setting $\phi(x_3) = x_3$. Then, we have $\psi(\phi(x_i)) = f_i$ for $i = 1, 3$, and so

$$\phi^{-1} \circ \psi^{-1} \circ F = (x_1, (\phi^{-1} \circ \psi^{-1})(f_2), x_3). \quad (7.1)$$

Since ϕ and ψ are elements of $T_3(k)$, it follows that F belongs to $T_3(k)$. This proves (i).

Next, we show (ii). Since $a \neq 0$ and $\deg_{\mathbf{w}} p \leq w_3$, we know that $\deg_{\mathbf{w}} f_3 = w_3$, and $f_3^{\mathbf{w}}$ depends on x_3 . If $\deg_{\mathbf{w}} F = |\mathbf{w}|$, then we have $\text{mdeg}_{\mathbf{w}} F = \mathbf{w}_{\sigma}$ for some $\sigma \in \mathfrak{S}_3$ by Theorem 3.3 (ii). Since $w_3 = \deg_{\mathbf{w}} f_3 = w_{\sigma(3)}$, we may assume that $\sigma(3) = 3$. Then, $G = (x_{\sigma(1)}, x_{\sigma(2)}, x_3)$ satisfies the required conditions. Assume that $\deg_{\mathbf{w}} F > |\mathbf{w}|$. Then, we have $\deg_{\mathbf{w}} f_1 + \deg_{\mathbf{w}} f_2 > w_1 + w_2$. If $f_2^{\mathbf{w}}$ belongs to $k[x_1, x_2]$, then the conditions (a) through (d) before Theorem 6.2 are fulfilled. In this case, the assertion follows from Theorem 6.2 (iii). Hence, we may assume that $f_2^{\mathbf{w}}$ does not belong to $k[x_1, x_2]$. By (7.1), we have

$$(\phi^{-1} \circ \psi^{-1})(f_2) = bx_2 + q(x_1, x_3)$$

for some $b \in k^{\times}$ and $q(x_1, x_3) \in k[x_1, x_3]$. Write

$$q(x_1, x_3) = q_1(x_1) + x_3 q_2(x_1, x_3),$$

where $q_1(x_1) \in k[x_1]$ and $q_2(x_1, x_3) \in k[x_1, x_3]$. Set

$$h_1 = b\phi(x_2) + q_1(f_1), \quad h_2 = f_3 q_2(f_1, f_3).$$

Then, h_1 belongs to $k[x_1, x_2]$, h_2 belongs to $k[f_1, f_3]$, and

$$f_2 = (\psi \circ \phi)(bx_2 + q(x_1, x_3)) = b\phi(x_2) + q(f_1, f_3) = h_1 + h_2.$$

Since $k[f_1, f_2, f_3] = k[f_1, h_1, f_3]$, and f_1 and h_1 belong to $k[x_1, x_2]$, we know that $k[f_1, h_1] = k[x_1, x_2]$. By the remark before Corollary 6.3, there exists $(g_1, g_2) \in E_2^{\mathbf{v}}(\kappa)$ such that $(g_1, g_2) \sim_{\mathbf{v}} (f_1, h_1)$, where $\mathbf{v} := (w_1, w_2)$. If $\deg_{\mathbf{w}} h_1 = \deg_{\mathbf{w}} f_2$, then $G \sim_{\mathbf{w}} F$ holds for $G := (g_1, g_2, x_3) \in E_3^{\mathbf{w}}(\kappa)$. Assume that $\deg_{\mathbf{w}} h_1 \neq \deg_{\mathbf{w}} f_2$. Then, we have $h_2 \neq 0$. Hence, $h_2^{\mathbf{w}} = f_3^{\mathbf{w}} q_2(f_1, f_3)^{\mathbf{w}}$ depends on x_3 . Since $f_2^{\mathbf{w}}$ does not belong to $k[x_1, x_2]$ by assumption, and h_1 is an element of $k[x_1, x_2]$ with $\deg_{\mathbf{w}} h_1 \neq \deg_{\mathbf{w}} f_2$, it follows that $\deg_{\mathbf{w}} f_2 = \deg_{\mathbf{w}} h_2$ and $\deg_{\mathbf{w}} f_2 > \deg_{\mathbf{w}} h_1 = \deg_{\mathbf{w}} g_2$. We claim that

$\deg_{\mathbf{w}} h_2$ belongs to $\mathbf{N}_0 \deg_{\mathbf{w}} f_1 + \mathbf{N}_0 \deg_{\mathbf{w}} f_3$. In fact, since $f_1^{\mathbf{w}} \in k[x_1, x_2] \setminus k$ and $f_3^{\mathbf{w}} \in k[\mathbf{x}] \setminus k[x_1, x_2]$ are algebraically independent over k , we have $k[f_1, f_3]^{\mathbf{w}} = k[f_1^{\mathbf{w}}, f_3^{\mathbf{w}}]$ by Corollary 2.3 (iii). Hence, we may write

$$\deg_{\mathbf{w}} f_2 = \deg_{\mathbf{w}} h_2 = l_1 \deg_{\mathbf{w}} f_1 + l_3 \deg_{\mathbf{w}} f_3 = l_1 \deg_{\mathbf{w}} f_1 + l_3 w_3,$$

where $l_1, l_3 \in \mathbf{N}_0$. Define $G \in E_3(\kappa)$ by $G = (g_1, g_2 + g_1^{l_1} x_3^{l_3}, x_3)$. Then, G belongs to $E_3^{\mathbf{w}}(\kappa)$ and satisfies $G \sim_{\mathbf{w}} F$, since $\deg_{\mathbf{w}} f_2 > \deg_{\mathbf{w}} g_2$. This proves (ii). \square

We note that Proposition 7.2 can be proved without using Theorem 7.1, since we can directly verify that f_1 is a coordinate of $k[x_1, x_2]$ over k as follows. Write $F^{-1} \circ \psi = (g_1, g_2, g_3)$. Then, we have

$$x_i = \psi^{-1}(F(g_i)) = g_i(\psi^{-1}(f_1), \psi^{-1}(f_2), \psi^{-1}(f_3)) = g_i(f_1, \psi^{-1}(f_2), x_3)$$

for $i = 1, 2, 3$. Let f'_2 be the element of $k[x_1, x_2]$ obtained from $\psi^{-1}(f_2)$ by the substitution $x_3 \mapsto 0$. Then, we have $x_i = g_i(f_1, f'_2, 0)$ for $i = 1, 2$. Hence, we get $k[f_1, f'_2] = k[x_1, x_2]$.

Now, let us prove Theorems 1.6 and 1.7. First, we show Theorem 1.6 and the case (1) of Theorem 1.7. By replacing k with the field of fractions of k , we may assume that k is a field. Take any $F \in \text{Aut}_k k[\mathbf{x}]$ such that at least two of $\deg_{\mathbf{w}} f_i$'s are not greater than $\max\{w_1, w_2, w_3\}$. We show that F belongs to $T_3(k)$ and $\text{mdeg}_{\mathbf{w}} F$ belongs to $|E_3^{\mathbf{w}}|$. By changing the indices of f_i 's, w_i 's and x_i 's if necessary, we may assume that F and \mathbf{w} satisfy (3.1). Then, $\deg_{\mathbf{w}} f_i \leq w_3$ holds for $i = 1, 2$. When f_1 and f_2 belong to $k[x_1, x_2]$, we have $k[f_1, f_2] = k[x_1, x_2]$. Since $\text{Aut}_k k[x_1, x_2] = T_2(k)$, this implies that F belongs to $T_3(k)$. Moreover, $\text{mdeg}_{\mathbf{w}} F$ belongs to $|E_3^{\mathbf{w}}|$ by Corollary 6.3 (i). Thus, we may assume that f_1 or f_2 does not belong to $k[x_1, x_2]$. If $\deg_{\mathbf{w}} f_1 < w_3$, then f_1 belongs to $k[x_1, x_2]$. Hence, f_2 does not belong to $k[x_1, x_2]$. Since $\deg_{\mathbf{w}} f_2 \leq w_3$, we may write $f_2 = ax_3 + p$, where $a \in k^\times$, and $p \in k[x_1, x_2]$ is such that $\deg_{\mathbf{w}} p \leq w_3$. Thus, the assertion follows from Proposition 7.2 (i) and (ii). The same holds when $\deg_{\mathbf{w}} f_2 < w_3$. So assume that $\deg_{\mathbf{w}} f_i = w_3$ for $i = 1, 2$. Then, we may write $f_i = a_i x_3 + p_i$ for $i = 1, 2$, where $a_i \in k$, and $p_i \in k[x_1, x_2]$ is such that $\deg_{\mathbf{w}} p_i \leq w_3$. Since f_1 or f_2 does not belong to $k[x_1, x_2]$, we may assume that $a_2 \neq 0$. Then, $f' := f_1 - a_1 a_2^{-1} f_2 = p_1 - a_1 a_2^{-1} p_2$ belongs to $k[x_1, x_2]$. Hence, $F' := (f', f_3, f_2)$ belongs to $T_3(k)$ by Proposition 7.2 (i), and thus so does F . Take any commutative ring κ . Then, there exists $(g_1, g_2, x_3) \in E_3^{\mathbf{w}}(\kappa)$ such that $(g_1, g_2, x_3) \sim_{\mathbf{w}} F'$ by Proposition 7.2 (ii). By the choice of p_1 and p_2 , we have $\deg_{\mathbf{w}} g_1 = \deg_{\mathbf{w}} f' \leq w_3$. Define $G \in E_3(\kappa)$ by $G = (g_1 + x_3, x_3, g_2)$ if $\deg_{\mathbf{w}} g_1 < w_3$, and by $G = (g_1, x_3, g_2)$ if $\deg_{\mathbf{w}} g_1 = w_3$. Then, G is an

element of $E_3^{\mathbf{w}}(\kappa)$ such that $G \sim_{\mathbf{w}} F$. Therefore, $\deg_{\mathbf{w}} F$ belongs to $|E_3^{\mathbf{w}}|$. This completes the proof of Theorem 1.6 and the case (1) of Theorem 1.7.

Next, we prove the case (2) of Theorem 1.7. Without loss of generality, we may assume that $w_1 \leq w_2 \leq w_3$ as before. Then, the conditions in (2) implies that

$$\deg_{\mathbf{w}} f_1 < w_3 \quad \text{and} \quad \deg_{\mathbf{w}} f_2 < w_3 + \deg_{\mathbf{w}} f_1 < 2w_3. \quad (7.2)$$

Hence, f_1 belongs to $k[x_1, x_2]$. Thus, if f_2 belongs to $k[x_1, x_2]$, then F belongs to $T_3(k)$ as before. Assume that f_2 does not belong to $k[x_1, x_2]$. Since f_1 is a coordinate of $k[x_1, x_2]$ over k by Theorem 7.1, there exists $g \in k[x_1, x_2]$ such that $k[f_1, g] = k[x_1, x_2]$. Then, we have $k'[g, x_3] = k'[f_2, f_3]$, where $k' := k[f_1]$. Hence, there exists a coordinate $p = p(y, z)$ of the polynomial ring $k'[y, z]$ over k' such that $f_2 = p(g, x_3)$. Then, we have $\deg_z p = \deg_{x_3} f_2 \leq 1$, since $\deg_{\mathbf{w}} f_2 < 2w_3$ by (7.2). Since f_2 does not belong to $k[x_1, x_2]$ by assumption, we conclude that $\deg_z p = 1$. Write $p = h_1 z + h_0$, where $h_0, h_1 \in k'[y]$ with $h_1 \neq 0$. Then, (7.2) yields that

$$w_3 + \deg_{\mathbf{w}} f_1 > \deg_{\mathbf{w}} f_2 = \deg_{\mathbf{w}} (h_1(g)x_3 + h_0(g)) \geq \deg_{\mathbf{w}} h_1(g)x_3,$$

and so $\deg_{\mathbf{w}} h_1(g) < \deg_{\mathbf{w}} f_1$. We show that h_1 belongs to k' . Put $d = \deg_y h_1$. Take any integer $l > \deg_y h_0 - d$, and define $\mathbf{v} = (1, l) \in \mathbf{Z}^2$. Then, we have $\deg_{\mathbf{v}} h_1 z = d + l > \deg_{\mathbf{v}} h_0$, and so

$$p^{\mathbf{v}} = (h_1 z + h_0)^{\mathbf{v}} = (h_1 z)^{\mathbf{v}} = h_1^{\mathbf{v}} z = ay^d z,$$

where $a \in k' \setminus \{0\}$ is the leading coefficient of h_1 . Since p is a coordinate of $k'[y, z]$ over k' , we know that $d = 0$ by the remark after Theorem 4.1. Thus, h_1 belongs to k' . Since $\deg_{\mathbf{w}} h_1(g) < \deg_{\mathbf{w}} f_1$ as mentioned, it follows that h_1 belongs to k . Therefore, $f_2 = h_1 x_3 + h_0(g)$ has the same form as f_3 in Proposition 7.2. Since f_1 belong to $k[x_1, x_2]$, we conclude that F belongs to $T_3(k)$ by Proposition 7.2 (i). This completes the proof of the case (2) of Theorem 1.7.

8 Tameness of weighted multidegrees

In this section, we give two kinds of sufficient conditions for elements of $\text{mdeg}_{\mathbf{w}}(\text{Aut}_k k[\mathbf{x}])$ to belong to $|E_n^{\mathbf{w}}|$, which can be viewed as generalizations of Proposition 1.5.

Lemma 8.1. *Let κ be any commutative ring, and let $\mathbf{w} \in \Gamma^n$ and $d_i, e_i \in \Gamma$ for $i = 1, \dots, n$. Assume that there exist $\sigma, \tau \in \mathfrak{S}_n$ and $0 \leq r \leq n$ such that*

$$d_{\sigma(i)} \in \sum_{j=1}^{i-1} \mathbf{N}_0 d_{\sigma(j)} + \sum_{j=i+1}^n \mathbf{N}_0 e_{\tau(j)} \quad \text{and} \quad d_{\sigma(i)} \geq e_{\tau(i)} \quad (8.1)$$

for $i = 1, \dots, r$, and $d_{\sigma(i)} = e_{\tau(i)}$ for $i = r+1, \dots, n$. If $\text{mdeg}_{\mathbf{w}} \psi = (e_1, \dots, e_n)$ for some $\psi \in \text{Aut}_{\kappa}^{\mathbf{w}} \kappa[\mathbf{x}]$, then there exists $\phi \in E_n(\kappa)$ such that $\psi \circ \phi$ belongs to $\text{Aut}_{\kappa}^{\mathbf{w}} \kappa[\mathbf{x}]$ and $\text{mdeg}_{\mathbf{w}} \psi \circ \phi = (d_1, \dots, d_n)$.

Proof. Set $s = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Then, it suffices to show that $\psi \circ (\phi \circ s^{-1})$ belongs to $\text{Aut}_{\kappa}^{\mathbf{w}} \kappa[\mathbf{x}]$ and $\text{mdeg}_{\mathbf{w}} \psi \circ (\phi \circ s^{-1}) = (d_1, \dots, d_n)$ for some $\phi \in E_n(\kappa)$, since ϕ belongs to $E_n(\kappa)$ if and only if so does $\phi \circ s^{-1}$. Note that $\psi \circ \phi \circ s^{-1}$ belongs to $\text{Aut}_{\kappa}^{\mathbf{w}} \kappa[\mathbf{x}]$ if and only if so does $\psi \circ \phi$, and $\text{mdeg}_{\mathbf{w}} \psi \circ \phi \circ s^{-1} = (d_1, \dots, d_n)$ if and only if $\text{mdeg}_{\mathbf{w}} \psi \circ \phi = (d_{\sigma(1)}, \dots, d_{\sigma(n)})$. Hence, we are reduced to proving that $\psi \circ \phi$ belongs to $\text{Aut}_{\kappa}^{\mathbf{w}} \kappa[\mathbf{x}]$ and $\text{mdeg}_{\mathbf{w}} \psi \circ \phi = (d_{\sigma(1)}, \dots, d_{\sigma(n)})$ for some $\phi \in E_n(\kappa)$. Therefore, we may assume that $\sigma = \text{id}$ by changing the indices of d_1, \dots, d_n if necessary. Next, set $t = (x_{\tau(1)}, \dots, x_{\tau(n)})$. Then, it suffices to show that $\psi \circ (t \circ \phi)$ belongs to $\text{Aut}_{\kappa}^{\mathbf{w}} \kappa[\mathbf{x}]$ and $\text{mdeg}_{\mathbf{w}} \psi \circ (t \circ \phi) = (d_1, \dots, d_n)$ for some $\phi \in E_n(\kappa)$ similarly. Since $\text{mdeg}_{\mathbf{w}} \psi \circ t = (e_{\tau(1)}, \dots, e_{\tau(n)})$, we may assume that $\tau = \text{id}$ by replacing ψ with $\psi \circ t$ and changing the indices of e_1, \dots, e_n if necessary.

We prove the lemma by induction on r . When $r = 0$, we have $d_i = e_i$ for each i . Since ψ is an element of $\text{Aut}_{\kappa}^{\mathbf{w}} \kappa[\mathbf{x}]$, the assertion holds for $\phi = \text{id}_{\kappa[\mathbf{x}]}$. Assume that $r \geq 1$. Then, the assumption of the lemma is satisfied even if (d_1, \dots, d_n) is replaced by $(d_1, \dots, d_{r-1}, e_r, \dots, e_n)$, since (8.1) holds for any $i < r$, and $d_i = e_i$ for $i \geq r$. Since r is reduced by one in this case, there exists $\phi' \in E_n(\kappa)$ such that $\psi \circ \phi'$ belongs to $\text{Aut}_{\kappa}^{\mathbf{w}} \kappa[\mathbf{x}]$ and

$$\text{mdeg}_{\mathbf{w}} \psi \circ \phi' = (d_1, \dots, d_{r-1}, e_r, \dots, e_n) \quad (8.2)$$

by induction assumption. By (8.1) with $i = r$, we have $d_r \geq e_r$ and

$$d_r = a_1 d_1 + \dots + a_{r-1} d_{r-1} + a_{r+1} e_{r+1} + \dots + a_n e_n$$

for some $a_j \in \mathbf{N}_0$ for each j . Set $f = (\psi \circ \phi')(x_1^{a_1} \dots x_n^{a_n})$ and $g = (\psi \circ \phi')(x_r)$, where $a_r = 0$. Then, we have $\deg_{\mathbf{w}} f = d_r$ and $\deg_{\mathbf{w}} g = e_r$ in view of (8.2). Since $\psi \circ \phi'$ is an element of $\text{Aut}_{\kappa}^{\mathbf{w}} \kappa[\mathbf{x}]$, we see that $f^{\mathbf{w}}$ and $g^{\mathbf{w}}$ are nonzero divisors of $\kappa[\mathbf{x}]$. Define $\phi'' \in E_n(\kappa)$ by $\phi''(x_r) = x_r + \alpha x_1^{a_1} \dots x_n^{a_n}$ and $\phi''(x_i) = x_i$ for each $i \neq r$, where $\alpha = 1$ if $d_r > e_r$, and $\alpha = 0$ if $d_r = e_r$. Then, we have $(\psi \circ \phi' \circ \phi'')(x_r) = g + \alpha f$. Since $\deg_{\mathbf{w}} g = e_r$ and $\deg_{\mathbf{w}} f = d_r$, we get $\deg_{\mathbf{w}}(\psi \circ \phi' \circ \phi'')(x_r) = d_r$ by the definition of α . Moreover, $(\psi \circ \phi' \circ \phi'')(x_r)^{\mathbf{w}}$

is equal to $f^{\mathbf{w}}$ or $g^{\mathbf{w}}$, and hence is a nonzero divisor of $\kappa[\mathbf{x}]$. If $i \neq r$, then we have $(\psi \circ \phi' \circ \phi'')(x_i) = (\psi \circ \phi')(x_i)$, for which $(\psi \circ \phi')(x_i)^{\mathbf{w}}$ is a nonzero divisor of $\kappa[\mathbf{x}]$. Thus, $\psi \circ \phi' \circ \phi''$ belongs to $\text{Aut}_{\kappa}^{\mathbf{w}} \kappa[\mathbf{x}]$. Moreover, we have

$$\text{mdeg}_{\mathbf{w}} \psi \circ \phi' \circ \phi'' = (d_1, \dots, d_r, e_{r+1}, \dots, e_n)$$

by (8.2). Therefore, the assertion holds for $\phi = \phi' \circ \phi''$. \square

Let us discuss the case of $n = 3$. For $\mathbf{w} \in \Gamma^3$, $d_1, d_2, d_3 \in \Gamma$ and $\sigma, \tau \in \mathfrak{S}_3$, consider the following conditions:

- (1) $d_{\sigma(i)} \geq w_{\tau(i)}$ for $i = 1, 2, 3$.
- (2) $d_{\sigma(1)}, d_{\sigma(2)}$ and $d_{\sigma(3)}$ belong to $\mathbf{N}_0 w_{\tau(2)} + \mathbf{N}_0 w_{\tau(3)}$, $\mathbf{N}_0 d_{\sigma(1)} + \mathbf{N}_0 w_{\tau(3)}$ and $\mathbf{N}_0 d_{\sigma(1)} + \mathbf{N}_0 d_{\sigma(2)}$, respectively.
- (3) $d_{\sigma(i)} \geq w_{\tau(i)}$ for $i = 1, 2$ and $d_{\sigma(3)} = w_{\tau(3)}$.
- (4) $d_{\sigma(1)}$ and $d_{\sigma(2)}$ belong to $\mathbf{N}_0 w_{\tau(2)} + \mathbf{N}_0 w_{\tau(3)}$ and $\mathbf{N}_0 d_{\sigma(1)} + \mathbf{N}_0 w_{\tau(3)}$, respectively.

If (1) and (2) are satisfied, then the assumption of Lemma 8.1 holds for $\psi = \text{id}_{\kappa[\mathbf{x}]}$ and $r = 3$. Hence, for any commutative ring κ , there exists $\phi \in E_3^{\mathbf{w}}(\kappa)$ such that $\text{mdeg}_{\mathbf{w}} \phi = (d_1, d_2, d_3)$ by Lemma 8.1. Therefore, (d_1, d_2, d_3) belongs to $|E_3^{\mathbf{w}}|$. The same holds when (3) and (4) are satisfied, since the assumption of Lemma 8.1 is fulfilled for $\psi = \text{id}_{\kappa[\mathbf{x}]}$ and $r = 2$.

With the aid of Theorems 1.4 and 1.6, we can derive the following theorem from Lemma 8.1.

Theorem 8.2. *Let $\mathbf{w} \in (\Gamma_+)^3$ and $(d_1, d_2, d_3) \in \text{mdeg}_{\mathbf{w}}(\text{Aut}_k k[\mathbf{x}])$ for $n = 3$ be such that*

$$d_1 \in \mathbf{N}_0 w_2 + \mathbf{N}_0 w_3, \quad d_2 \in \mathbf{N}_0 d_1 + \mathbf{N}_0 w_3, \quad d_3 \in \mathbf{N}_0 d_1 + \mathbf{N}_0 d_2. \quad (8.3)$$

If one of the following conditions holds, then (d_1, d_2, d_3) belongs to $|E_3^{\mathbf{w}}|$:

- (a) $d_1 \leq d_2$. (b) $d_2 \geq w_2$. (c) $d_2 = w_3$. (d) $d_1 \in (\mathbf{N}_0 w_3) \cup (\mathbf{N}_0 w_2 + \mathbf{N}_0 d_2)$.

Proof. Thanks to Theorem 1.6, we may assume that two of d_1, d_2 and d_3 are greater than $w := \max\{w_1, w_2, w_3\}$. Since $d_3 \neq 0$ belongs to $\mathbf{N}_0 d_1 + \mathbf{N}_0 d_2$ by (8.3), we have $d_3 \geq d_1$ or $d_3 \geq d_2$. Hence, we may assume that $d_3 > w$. Similarly, we may assume that $d_2 > w$ if (a) holds, and $d_1 > w$ otherwise. In the following, we check that (1) and (2), or (3) and (4) hold for some $\sigma, \tau \in \mathfrak{S}_3$. We note that (2) and (4) are clear from (8.3) if $\sigma = \tau = \text{id}$.

First, assume that (a) holds. Then, we have

$$d_i > w \geq w_j \quad \text{for } i = 2, 3 \quad \text{and} \quad j = 1, 2, 3. \quad (8.4)$$

Hence, if $d_1 \geq w_1$, then (1) holds for $\sigma = \tau = \text{id}$. Since (2) holds for $\sigma = \tau = \text{id}$ as mentioned, we may assume that $d_1 < w_1$. By Theorem 1.4, d_1 belongs to $C(\mathbf{w})$ or $\{w_1, w_2, w_3\}$. Hence, there exists $1 \leq i \leq 3$ such that $d_1 \geq w_i$ and $d_1 \in \sum_{j \neq i} \mathbf{N}_0 d_j$, or $d_1 = w_i$. Since $d_1 < w_1$, it follows that $d_1 \geq w_{\rho(2)}$ and $d_1 \in \mathbf{N}_0 w_{\rho(3)}$, or $d_1 = w_{\rho(3)}$ for some $\rho \in \{\text{id}, (2, 3)\}$. We show that (1) and (2) hold for $\sigma = (1, 2)$ and $\tau = \rho$ when $d_1 \geq w_{\rho(2)}$ and $d_1 \in \mathbf{N}_0 w_{\rho(3)}$. Since $d_{\sigma(2)} \geq w_{\rho(2)}$, we have (1) due to (8.4). By (8.3), $d_{\sigma(1)}$ belongs to $\mathbf{N}_0 d_1 + \mathbf{N}_0 w_3$. Since $d_1 \in \mathbf{N}_0 w_{\rho(3)}$ and $3 \in \{\rho(2), \rho(3)\}$, we have $\mathbf{N}_0 d_1 + \mathbf{N}_0 w_3 \subset \mathbf{N}_0 w_{\rho(2)} + \mathbf{N}_0 w_{\rho(3)}$. Thus, we get $d_{\sigma(1)} \in \mathbf{N}_0 w_{\rho(2)} + \mathbf{N}_0 w_{\rho(3)}$. Since $d_1 \in \mathbf{N}_0 w_{\rho(3)}$, we have $d_{\sigma(2)} \in \mathbf{N}_0 d_{\sigma(1)} + \mathbf{N}_0 w_{\rho(3)}$. Since $d_3 \in \mathbf{N}_0 d_1 + \mathbf{N}_0 d_2$ by (8.3), and $\sigma = (1, 2)$, we have $d_{\sigma(3)} \in \mathbf{N}_0 d_{\sigma(1)} + \mathbf{N}_0 d_{\sigma(2)}$. Therefore, (2) is satisfied. Next, we show that (3) and (4) hold for $\sigma = (1, 2, 3)$ and $\tau = \rho$ when $d_1 = w_{\rho(3)}$. Since $d_{\sigma(3)} = w_{\rho(3)}$, we have (3) due to (8.4). Since $\mathbf{N}_0 d_1 + \mathbf{N}_0 w_3 \subset \mathbf{N}_0 w_{\rho(2)} + \mathbf{N}_0 w_{\rho(3)}$ and $\mathbf{N}_0 d_1 + \mathbf{N}_0 d_2 = \mathbf{N}_0 w_{\rho(3)} + \mathbf{N}_0 d_{\sigma(1)}$, (4) follows from (8.3).

Next, assume that (a) does not hold. Then, we have $d_i > w \geq w_j$ for $i = 1, 3$ and $j = 1, 2, 3$ as remarked. Hence, if (b) is satisfied, then (1) and (2) hold for $\sigma = \tau = \text{id}$ as before. In the case of (c), (3) and (4) hold for $\sigma = (2, 3)$ and $\tau = \text{id}$, since $d_{\sigma(2)}$ belongs to $\mathbf{N}_0 d_1 + \mathbf{N}_0 d_2 = \mathbf{N}_0 d_1 + \mathbf{N}_0 w_3$. Finally, we consider the case (d). In view of (b) and (c), we may assume that $d_2 < w_2$ and $d_2 \neq w_3$. We claim that $d_2 \geq w_1$. In fact, if not, we have $d_2 < w_i$ for $i = 1, 2$. This implies that $d_2 = \deg_{\mathbf{w}} f$ for a coordinate f of $k[\mathbf{x}]$ over k belonging to $k[x_3]$, and so $d_2 = w_3$, a contradiction. Hence, (1) holds for $\sigma = \text{id}$ and $\tau = (1, 2)$, and for $\sigma = (1, 2)$ and $\tau = (2, 3)$. If d_1 belongs to $\mathbf{N}_0 w_3$, then (2) holds for $\sigma = \text{id}$ and $\tau = (1, 2)$ by (8.3). We check that (2) holds for $\sigma = (1, 2)$ and $\tau = (2, 3)$ when $d_1 \notin \mathbf{N}_0 w_3$. Since $d_{\sigma(1)}$ belongs to $\mathbf{N}_0 d_1 + \mathbf{N}_0 w_3$ by (8.3), and (a) does not hold by assumption, $d_{\sigma(1)}$ belongs to $\mathbf{N}_0 w_3$, and hence to $\mathbf{N}_0 w_{\tau(2)} + \mathbf{N}_0 w_{\tau(3)}$. Since $d_1 \notin \mathbf{N}_0 w_3$, we know by (d) that $d_{\sigma(2)} = d_1$ belongs to $\mathbf{N}_0 w_2 + \mathbf{N}_0 d_2 = \mathbf{N}_0 d_{\sigma(1)} + \mathbf{N}_0 w_{\tau(3)}$. Since $d_3 \in \mathbf{N}_0 d_1 + \mathbf{N}_0 d_2$ by (8.3), and $\sigma = (1, 2)$, we have $d_{\sigma(3)} \in \mathbf{N}_0 d_{\sigma(1)} + \mathbf{N}_0 d_{\sigma(2)}$. Therefore, (2) is satisfied. \square

Next, we give another kind of generalization of Proposition 1.5. Assume that $n \geq 2$. Take any $d_1, \dots, d_n \in \Gamma_+$ and $\mathbf{w} \in (\Gamma_+)^n$. For $d \in \Gamma_+$, $1 \leq l \leq n$ and $2 \leq m \leq n$, consider the following conditions:

- (a) d_1, \dots, d_n belong to $\mathbf{N}d$.
- (b) $d = w_l$, or $d > w_l$ and d belongs to $\sum_{j \neq l} \mathbf{N}_0 w_j$.
- (c) d_m belongs to $\sum_{j=1}^{m-1} \mathbf{N}_0 d_j$.
- (d) If $l < m$, then $d_i \geq w_{i+1}$ for each $l \leq i < m$.

Then, we have the following lemma.

Lemma 8.3. *Let $\mathbf{w} \in (\Gamma_+)^n$ and $(d_1, \dots, d_n) \in \text{mdeg}_{\mathbf{w}}(\text{Aut}_k k[\mathbf{x}])$ for $n \geq 2$ be such that $w_1 \leq \dots \leq w_n$ and $d_1 \leq \dots \leq d_n$. If there exist $d \in \Gamma_+$, $1 \leq l \leq n$ and $2 \leq m \leq n$ which satisfy (a) through (d), then (d_1, \dots, d_n) belongs to $|\mathbf{E}_n^{\mathbf{w}}|$.*

Proof. We remark that $d_i \geq w_i$ for $i = 1, \dots, n$ by Theorem 3.3 (i). Take any commutative ring κ . We define $g \in \kappa[\mathbf{x}]$ by $g = x_l$ if $d = w_l$. If $d \neq w_l$, then we have $d > w_l$ and $d = \sum_{j \neq l} a_j w_j$ for some $a_j \in \mathbf{N}_0$ by (b). In this case, we define $g = x_l + \prod_{j \neq l} x_j^{a_j}$. Then, $g^{\mathbf{w}}$ is a nonzero divisor of $\kappa[\mathbf{x}]$ and $\deg_{\mathbf{w}} g = d$ in either case. By (a), we may write $d_i = e_i d$ for $i = 1, \dots, n$, where $e_i \in \mathbf{N}$. We define $\phi \in \mathbf{E}_n(\kappa)$ by

$$\phi(x_i) = \begin{cases} g & \text{if } i = m \\ x_i + \alpha_i g^{e_i} & \text{if } i < \min\{l, m\} \text{ or } i > \max\{l, m\} \\ x_{i-1} + \beta_i g^{e_i} & \text{if } m < i \leq l \\ x_{i+1} + \gamma_i g^{e_i} & \text{if } l \leq i < m, \end{cases}$$

where $\alpha_i = 1$ if $d_i > w_i$, and $\alpha_i = 0$ otherwise, where $\beta_i = 1$ if $d_i > w_{i-1}$, and $\beta_i = 0$ otherwise, and where $\gamma_i = 1$ if $d_i > w_{i+1}$, and $\gamma_i = 0$ otherwise. Then, each $\phi(x_i)^{\mathbf{w}}$ is a power of $g^{\mathbf{w}}$ or one of x_i , x_{i-1} and x_{i+1} . Hence, $\phi(x_i)^{\mathbf{w}}$ is a nonzero divisor of $\kappa[\mathbf{x}]$ for each i . We show that $\deg_{\mathbf{w}} \phi(x_i) = d_i$ for each $i \neq m$. This is clear in the cases where $\alpha_i = 1$, $\beta_i = 1$ and $\gamma_i = 1$, since $\deg_{\mathbf{w}} g^{e_i} = d_i$ is greater than w_i , w_{i-1} and w_{i+1} in the respective cases. If $\alpha_i = 0$, then we have $\phi(x_i) = x_i$ and $d_i \leq w_i$. Since $d_i \geq w_i$ as remarked, it follows that $\deg_{\mathbf{w}} \phi(x_i) = w_i = d_i$. If $\beta_i = 0$, then we have $\phi(x_i) = x_{i-1}$ and $d_i \leq w_{i-1}$. Since $d_i \geq w_i \geq w_{i-1}$, we get $\deg_{\mathbf{w}} \phi(x_i) = w_{i-1} = d_i$. If $\gamma_i = 0$, then we have $\phi(x_i) = x_{i+1}$ and $d_i \leq w_{i+1}$. Since $d_i \geq w_{i+1}$ by (d), we get $\deg_{\mathbf{w}} \phi(x_i) = w_{i+1} = d_i$. Thus, $\deg_{\mathbf{w}} \phi(x_i) = d_i$ holds for each $i \neq m$. By (c), we may write $d_m = \sum_{j=1}^{m-1} c_j d_j$, where $c_j \in \mathbf{N}_0$ for each j . Set $f = \phi(x_1^{c_1} \dots x_{m-1}^{c_{m-1}})$. Then, $f^{\mathbf{w}}$ is a nonzero divisor of $\kappa[\mathbf{x}]$ and $\deg_{\mathbf{w}} f = d_m = e_m d \geq d$. Define $\psi \in \mathbf{E}_n(\kappa)$ by $\psi(x_m) = x_m + \delta x_1^{c_1} \dots x_{m-1}^{c_{m-1}}$ and $\psi(x_i) = x_i$ for each $i \neq m$, where $\delta = 1$ if $d_m > d$, and $\delta = 0$ if $d_m = d$. Then, we have $(\phi \circ \psi)(x_m) = g + \delta f$. Since $\deg_{\mathbf{w}} g = d$ and $\deg_{\mathbf{w}} f = d_m$, we get $\deg_{\mathbf{w}}(\phi \circ \psi)(x_m) = d_m$ by the definition of δ . Moreover, $(\phi \circ \psi)(x_m)^{\mathbf{w}}$ is equal to $f^{\mathbf{w}}$ or $g^{\mathbf{w}}$, and hence is a nonzero divisor of $\kappa[\mathbf{x}]$. If $i \neq m$, then we have $(\phi \circ \psi)(x_i) = \phi(x_i)$, for which $\phi(x_i)^{\mathbf{w}}$ is a nonzero divisor of $\kappa[\mathbf{x}]$ and $\deg_{\mathbf{w}} \phi(x_i) = d_i$. Thus, $\phi \circ \psi$ is an element of $\mathbf{E}_n^{\mathbf{w}}(\kappa)$ and satisfies $\text{mdeg}_{\mathbf{w}} \phi \circ \psi = (d_1, \dots, d_n)$. Therefore, (d_1, \dots, d_n) belongs to $|\mathbf{E}_n^{\mathbf{w}}|$. \square

Let us discuss the case of $n = 3$. For $d_1, d_2, d_3, d \in \Gamma_+$ and $\mathbf{w} \in (\Gamma_+)^3$, consider the following conditions:

(A) d_1, d_2 and d_3 belong to $\mathbf{N}d$.

(B) d belongs to $\mathbf{N}_0 w_i + \mathbf{N}_0 w_3$ for some $i \in \{1, 2\}$, or $d \geq w_3$ and d belongs to $\mathbf{N} w_1 + \mathbf{N} w_2$.

The following theorem is a refinement of Lemma 8.3 in the case of $n = 3$. In fact, (a) is equivalent to (A). If (b) holds for some $1 \leq l \leq 3$, then we have (B). We have (c) for some $2 \leq m \leq 3$ if and only if $d_2 \in \mathbf{N} d_1$ or $d_3 \in \mathbf{N}_0 d_1 + \mathbf{N}_0 d_2$.

Theorem 8.4. *Let $\mathbf{w} \in (\Gamma_+)^3$ and $(d_1, d_2, d_3) \in \text{mdeg}_{\mathbf{w}}(\text{Aut}_k k[\mathbf{x}])$ for $n = 3$ be such that $w_1 \leq w_2 \leq w_3$, $d_1 \leq d_2 \leq d_3$, and $d_2 \in \mathbf{N} d_1$ or $d_3 \in \mathbf{N}_0 d_1 + \mathbf{N}_0 d_2$. If (A) and (B) hold for some $d \in \Gamma_+$, then (d_1, d_2, d_3) belongs to $|\mathbf{E}_3^{\mathbf{w}}|$.*

Proof. Thanks to Theorem 1.6, we may assume that two of d_1 , d_2 and d_3 are greater than $\max\{w_1, w_2, w_3\}$. Then, we have $d_3 \geq d_2 > w_3 \geq w_2 \geq w_1$. Since $d \neq 0$, we have $d \geq w_1$ by (B).

First, assume that $d_1 < w_2$. Then, we have $d_1 = \deg_{\mathbf{w}} f$ for some coordinate f of $k[\mathbf{x}]$ over k belonging to $k[x_1]$. Hence, we know that $d_1 = w_1$. By (A), we may write $d_i = e_i d$ for $i = 1, 2, 3$, where $e_i \in \mathbf{N}$. Since $d \geq w_1$ as mentioned, we get $d_1 = d = w_1$. Take any commutative ring κ , and define $\phi \in \mathbf{E}_3^{\mathbf{w}}(\kappa)$ by $\phi(x_1) = x_1$ and $\phi(x_i) = x_i + x_1^{e_i}$ for $i = 2, 3$. Then, we have $\text{mdeg}_{\mathbf{w}} \phi = (d_1, d_2, d_3)$, since $d_i > w_i$ for $i = 2, 3$. Therefore, (d_1, d_2, d_3) belongs to $|\mathbf{E}_3^{\mathbf{w}}|$.

Next, assume that $d_1 \geq w_2$. We show that (d_1, d_2, d_3) belongs to $|\mathbf{E}_3^{\mathbf{w}}|$ using Lemma 8.3. Since $d_2 \in \mathbf{N} d_1$ or $d_3 \in \mathbf{N}_0 d_1 + \mathbf{N}_0 d_2$ by assumption, (c) holds for some $2 \leq m \leq 3$. Since $d_1 \geq w_2$ and $d_2 > w_3$, (d) holds for any $1 \leq l \leq 3$. If d belongs to $\mathbf{N} w_1$, then d_1, d_2 and d_3 belong to $\mathbf{N} w_1$. When this is the case, (a) and (b) are satisfied if we take d to be w_1 . Assume that d does not belong to $\mathbf{N} w_1$. If the first part of (B) holds, then d belongs to $\mathbf{N}_0 w_1 + \mathbf{N} w_3$ or $\mathbf{N}_0 w_2 + \mathbf{N}_0 w_3$. In the first case, we have $d \geq w_3 \geq w_2$, and so (b) holds for $l = 2$. Since $d \geq w_1$ as mentioned, (b) holds for $l = 1$ in the second case. The last part of (B) implies that (b) holds for $l = 3$. Thus, (B) implies (b). Clearly, (A) implies (a). Therefore, we conclude that (d_1, d_2, d_3) belongs to $|\mathbf{E}_3^{\mathbf{w}}|$ by Lemma 8.3. \square

9 Shestakov-Umirbaev reductions

The goal of this section is to prove Theorems 1.9 and 1.10. To prove Theorem 1.9, we use the generalized Shestakov-Umirbaev theory [12], [13]. For the convenience of the reader, we give a short introduction to this theory. Assume that $n = 3$. For $F, G \in \text{Aut}_k k[\mathbf{x}]$, we say that the pair (F, G) satisfies the *Shestakov-Umirbaev condition* for the weight \mathbf{w} if the following conditions hold (cf. [13]).

- (SU1) $g_1 = f_1 + af_3^2 + cf_3$ and $g_2 = f_2 + bf_3$ for some $a, b, c \in k$, and $g_3 - f_3$ belongs to $k[g_1, g_2]$.
- (SU2) $\deg_{\mathbf{w}} f_1 \leq \deg_{\mathbf{w}} g_1$ and $\deg_{\mathbf{w}} f_2 = \deg_{\mathbf{w}} g_2$.
- (SU3) $(g_1^{\mathbf{w}})^2 \approx (g_2^{\mathbf{w}})^s$ for some odd number $s \geq 3$.
- (SU4) $\deg_{\mathbf{w}} f_3 \leq \deg_{\mathbf{w}} g_1$, and $f_3^{\mathbf{w}}$ does not belong to $k[g_1^{\mathbf{w}}, g_2^{\mathbf{w}}]$.
- (SU5) $\deg_{\mathbf{w}} g_3 < \deg_{\mathbf{w}} f_3$.
- (SU6) $\deg_{\mathbf{w}} g_3 < \deg_{\mathbf{w}} g_1 - \deg_{\mathbf{w}} g_2 + \deg_{\mathbf{w}} dg_1 \wedge dg_2$.

Here, we recall that $f_1 \approx f_2$ denotes that f_1 and f_2 are linearly dependent over k for each $f_1, f_2 \in k[\mathbf{x}]$. For each $F \in \text{Aut}_k k[\mathbf{x}]$ and $\sigma \in \mathfrak{S}_3$, we define $F_{\sigma} = (f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)})$. We say that $F \in \text{Aut}_k k[\mathbf{x}]$ admits a *Shestakov-Umirbaev reduction* for the weight \mathbf{w} if there exist $\sigma \in \mathfrak{S}_3$ and $G \in \text{Aut}_k k[\mathbf{x}]$ such that (F_{σ}, G_{σ}) satisfies the Shestakov-Umirbaev condition for the weight \mathbf{w} .

The following theorem is the main result of [13].

Theorem 9.1 ([13, Theorem 2.1]). *Assume that k is a field of characteristic zero. If $\deg_{\mathbf{w}} F > |\mathbf{w}|$ holds for $F \in T_3(k)$ and $\mathbf{w} \in (\Gamma_+)^3$, then F admits an elementary reduction or a Shestakov-Umirbaev reduction for the weight \mathbf{w} .*

Thanks to Theorem 9.1, the proof of Theorem 1.9 is reduced to the proof of the following lemma.

Lemma 9.2. *Assume that k is a field of characteristic zero, and \mathbf{w} is an element of $(\Gamma_+)^3$. Then, no element of $S(\mathbf{w}, k)$ admits a Shestakov-Umirbaev reduction for the weight \mathbf{w} .*

We note that, if (F, G) satisfies the Shestakov-Umirbaev condition for the weight \mathbf{w} , then (F, G) satisfies the “weak Shestakov-Umirbaev condition” for the weight \mathbf{w} , and has the following properties (cf. [13, Theorem 4.2]). Here, we regard Γ as a subgroup of $\mathbf{Q} \otimes_{\mathbf{Z}} \Gamma$ which has a structure of totally ordered additive group induced from Γ :

- (P1) $(g_1^{\mathbf{w}})^2 \approx (g_2^{\mathbf{w}})^s$ for some odd number $s \geq 3$. Hence, $\delta := (1/2) \deg_{\mathbf{w}} g_2$ belongs to Γ .
- (P5) If $\deg_{\mathbf{w}} f_1 < \deg_{\mathbf{w}} g_1$, then $s = 3$, $g_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$, $\deg_{\mathbf{w}} f_3 = (3/2)\delta$ and

$$\deg_{\mathbf{w}} f_1 \geq \frac{5}{2}\delta + \deg_{\mathbf{w}} dg_1 \wedge dg_2.$$

- (P6) $\deg_{\mathbf{w}} G < \deg_{\mathbf{w}} F$.
- (P7) $\deg_{\mathbf{w}} f_2 < \deg_{\mathbf{w}} f_1$, $\deg_{\mathbf{w}} f_3 \leq \deg_{\mathbf{w}} f_1$, and $\delta < \deg_{\mathbf{w}} f_i \leq s\delta$ for $i = 1, 2, 3$.

Now, let us prove Lemma 9.2 by contradiction. Suppose that F admits a Shestakov-Umirbaev reduction for the weight \mathbf{w} for some $F \in S(\mathbf{w}, k)$. Then, there exist $\sigma \in \mathfrak{S}_3$ and $G \in \text{Aut}_k k[\mathbf{x}]$ such that (F_σ, G_σ) satisfies the Shestakov-Umirbaev condition for the weight \mathbf{w} . Moreover, we have $f_3 = \alpha x_3 + p$ for some $\alpha \in k^\times$ and $p \in k[x_1, x_2]$ with $\deg_{\mathbf{w}} p \leq w_3$, and so $\deg_{\mathbf{w}} f_3 = w_3$.

First, we consider the case of $\sigma(1) = 3$. In this case, we have $\deg_{\mathbf{w}} f_{\sigma(1)} = \deg_{\mathbf{w}} f_3 = w_3$. Since $\deg_{\mathbf{w}} f_{\sigma(1)} > \deg_{\mathbf{w}} f_{\sigma(2)}$ by (P7), and $\deg_{\mathbf{w}} f_{\sigma(2)} = \deg_{\mathbf{w}} g_{\sigma(2)}$ by (SU2), it follows that $\deg_{\mathbf{w}} f_{\sigma(2)}$ and $\deg_{\mathbf{w}} g_{\sigma(2)}$ are less than w_3 . Hence, $f_{\sigma(2)}$ and $g_{\sigma(2)}$ belong to $k[x_1, x_2]$.

When $\deg_{\mathbf{w}} f_{\sigma(1)} = \deg_{\mathbf{w}} g_{\sigma(1)}$, we have $\deg_{\mathbf{w}} g_{\sigma(1)} = w_3$. Hence, $g_{\sigma(1)} - g_{\sigma(1)}^{\mathbf{w}}$ belongs to $k[x_1, x_2]$, since $\deg_{\mathbf{w}}(g_{\sigma(1)} - g_{\sigma(1)}^{\mathbf{w}}) < w_3$. By (SU3), $(g_{\sigma(1)}^{\mathbf{w}})^2 \approx (g_{\sigma(2)}^{\mathbf{w}})^s$ holds for some odd number $s \geq 3$. Since $g_{\sigma(2)}$ belongs to $k[x_1, x_2]$, it follows that $g_{\sigma(1)}^{\mathbf{w}}$ also belongs to $k[x_1, x_2]$. Thus, $g_{\sigma(1)}$ belongs to $k[x_1, x_2]$. Therefore, we can define $G' \in \text{Aut}_k k[x_1, x_2]$ by $G' = (g_{\sigma(1)}, g_{\sigma(2)})$. Since $g_{\sigma(1)}^{\mathbf{w}}$ and $g_{\sigma(2)}^{\mathbf{w}}$ are algebraically dependent over k , we have $\deg_{\mathbf{v}} G' > |\mathbf{v}|$ by Theorem 3.3 (ii), where $\mathbf{v} := (w_1, w_2)$. Hence, we have $g_{\sigma(1)}^{\mathbf{w}} \approx (g_{\sigma(2)}^{\mathbf{w}})^u$ or $g_{\sigma(2)}^{\mathbf{w}} \approx (g_{\sigma(1)}^{\mathbf{w}})^u$ for some $u \geq 1$ by Lemma 6.1. This contradicts that $(g_{\sigma(1)}^{\mathbf{w}})^2 \approx (g_{\sigma(2)}^{\mathbf{w}})^s$ with $s \geq 3$ an odd number.

When $\deg_{\mathbf{w}} f_{\sigma(1)} \neq \deg_{\mathbf{w}} g_{\sigma(1)}$, we have $\deg_{\mathbf{w}} f_{\sigma(1)} < \deg_{\mathbf{w}} g_{\sigma(1)}$ in view of (SU2). From (P5) and (SU2), it follows that

$$\deg_{\mathbf{w}} f_{\sigma(3)} = \frac{3}{2}\delta = \frac{3}{2}\deg_{\mathbf{w}} g_{\sigma(2)} = \frac{3}{4}\deg_{\mathbf{w}} f_{\sigma(2)},$$

and hence $4\deg_{\mathbf{w}} f_{\sigma(3)} = 3\deg_{\mathbf{w}} f_{\sigma(2)}$. Thus, we get $\deg_{\mathbf{w}} f_{\sigma(3)} < \deg_{\mathbf{w}} f_{\sigma(2)}$. Since $\deg_{\mathbf{w}} f_{\sigma(2)} < w_3$ as mentioned, it follows that $f_{\sigma(3)}$ belongs to $k[x_1, x_2]$. Hence, we can define $F' \in \text{Aut}_k k[x_1, x_2]$ by $F' = (f_{\sigma(2)}, f_{\sigma(3)})$. Since

$$w_3 + \deg_{\mathbf{v}} F' = \deg_{\mathbf{w}} f_{\sigma(1)} + \deg_{\mathbf{v}} F' = \deg_{\mathbf{w}} F_\sigma > \deg_{\mathbf{w}} G_\sigma \geq |\mathbf{w}| = |\mathbf{v}| + w_3$$

by (P6) and Theorem 3.3 (i), we have $\deg_{\mathbf{v}} F' > |\mathbf{v}|$. Thus, we know by Lemma 6.1 that $f_{\sigma(2)}^{\mathbf{w}} \approx (f_{\sigma(3)}^{\mathbf{w}})^u$ or $f_{\sigma(3)}^{\mathbf{w}} \approx (f_{\sigma(2)}^{\mathbf{w}})^u$ for some $u \geq 1$. This contradicts that $4\deg_{\mathbf{w}} f_{\sigma(3)} = 3\deg_{\mathbf{w}} f_{\sigma(2)}$.

Next, assume that $\sigma(1) \neq 3$. Due to (SU1), we can define $H \in \text{Aut}_k k[\mathbf{x}]$ by $H = (g_{\sigma(1)}, g_{\sigma(2)}, f_{\sigma(3)})$. In the following, we show that H and \mathbf{w} satisfy the conditions (a) through (d) before Theorem 6.2. Then, it follows that $g_{\sigma(1)}^{\mathbf{w}} \approx (g_{\sigma(2)}^{\mathbf{w}})^u$ or $g_{\sigma(2)}^{\mathbf{w}} \approx (g_{\sigma(1)}^{\mathbf{w}})^u$ for some $u \geq 1$ by Theorem 6.2 (i). Since $(g_{\sigma(1)}^{\mathbf{w}})^2 \approx (g_{\sigma(2)}^{\mathbf{w}})^s$ with $s \geq 3$ an odd number, we are led to a contradiction.

Since $(1/2)\deg_{\mathbf{w}} g_{\sigma(2)} = \delta < \deg_{\mathbf{w}} f_3 = w_3$ by (P7), we have $\deg_{\mathbf{w}} g_{\sigma(2)} < 2w_3$. This implies that $\deg_{x_3} g_{\sigma(2)}^{\mathbf{w}} \leq 1$. Since $(g_{\sigma(1)}^{\mathbf{w}})^2 \approx (g_{\sigma(2)}^{\mathbf{w}})^s$ with $s \geq 3$ an odd number, it follows that $\deg_{x_3} g_{\sigma(1)}^{\mathbf{w}} = \deg_{x_3} g_{\sigma(2)}^{\mathbf{w}} = 0$. Hence, $g_{\sigma(1)}^{\mathbf{w}}$

and $g_{\sigma(2)}^{\mathbf{w}}$ belong to $k[x_1, x_2]$, proving (a). We show that $f_{\sigma(3)} = \beta x_3 + q$ for some $\beta \in k^\times$ and $q \in k[x_1, x_2]$ with $\deg_{\mathbf{w}} q \leq w_3$. Then, we get (c) and (d) immediately. Since $\deg_{\mathbf{w}} g_{\sigma(3)} < \deg_{\mathbf{w}} f_{\sigma(3)}$ by (SU5), and $\deg_{\mathbf{w}} f_{\sigma(3)} = w_3$ by (c), we have

$$\begin{aligned} \deg_{\mathbf{w}} g_{\sigma(1)} + \deg_{\mathbf{w}} g_{\sigma(2)} &= \deg_{\mathbf{w}} G - \deg_{\mathbf{w}} g_{\sigma(3)} \\ &> \deg_{\mathbf{w}} G - \deg_{\mathbf{w}} f_{\sigma(3)} \geq |\mathbf{w}| - \deg_{\mathbf{w}} f_{\sigma(3)} = w_1 + w_2. \end{aligned}$$

Hence, (b) is also proved.

Since $\sigma(3) \neq 1$, we have $\sigma(2) = 3$ or $\sigma(3) = 3$. Recall that $f_3 = \alpha x_3 + p$ for some $\alpha \in k^\times$ and $p \in k[x_1, x_2]$ with $\deg_{\mathbf{w}} p \leq w_3$. Hence, the assertion is clear if $\sigma(3) = 3$. Assume that $\sigma(2) = 3$. Then, we have $\deg_{\mathbf{w}} g_{\sigma(2)} = \deg_{\mathbf{w}} f_{\sigma(2)} = \deg_{\mathbf{w}} f_3 = w_3$ by (SU2). Since $g_{\sigma(2)}^{\mathbf{w}}$ belongs to $k[x_1, x_2]$ as shown above, this implies that $g_{\sigma(2)}$ belongs to $k[x_1, x_2]$. By (SU1), there exists $b \in k$ such that

$$g_{\sigma(2)} = f_{\sigma(2)} + b f_{\sigma(3)} = \alpha x_3 + p + b f_{\sigma(3)}.$$

Since $g_{\sigma(2)}$ and p belong to $k[x_1, x_2]$ and $\alpha \neq 0$, it follows that $b \neq 0$ and

$$f_{\sigma(3)} = -\alpha b^{-1} x_3 + b^{-1} (g_{\sigma(2)} - p).$$

Since $g_{\sigma(2)}$ and p are elements of $k[x_1, x_2]$ with $\deg_{\mathbf{w}} g_{\sigma(2)} = w_3$ and $\deg_{\mathbf{w}} p \leq w_3$, we see that $f_{\sigma(3)}$ has the required form. This completes the proof of Lemma 9.2, and thereby completing the proof of Theorem 1.9.

The rest of this section is devoted to the proof of Theorem 1.10. To prove (ii) of this theorem, we need the following version of the Shestakov-Umirbaev inequality (see [13, Section 3] for detail). Let $S = \{f, g\}$ be a subset of $k[\mathbf{x}]$ such that f and g are algebraically independent over k , and p a nonzero element of $k[S]$. Then, we can uniquely express $p = \sum_{i,j} c_{i,j} f^i g^j$, where $c_{i,j} \in k$ for each $i, j \in \mathbf{N}_0$. We define $\deg_{\mathbf{w}}^S p$ to be the maximum among $\deg_{\mathbf{w}} f^i g^j$ for $i, j \in \mathbf{N}_0$ with $c_{i,j} \neq 0$. We note that, if $p^{\mathbf{w}}$ does not belong to $k[f^{\mathbf{w}}, g^{\mathbf{w}}]$, then $\deg_{\mathbf{w}}^S p$ is greater than $\deg_{\mathbf{w}} p$.

With the notation and assumption above, the following lemma holds (see [13, Lemmas 3.2 (i) and 3.3 (ii)] for the proof).

Lemma 9.3. *Assume that k is a field of characteristic zero. If $\deg_{\mathbf{w}}^S p$ is greater than $\deg_{\mathbf{w}} p$, then there exist $l, m \in \mathbf{N}$ with $\gcd(l, m) = 1$ such that $(g^{\mathbf{w}})^l \approx (f^{\mathbf{w}})^m$ and*

$$\deg_{\mathbf{w}} p \geq m \deg_{\mathbf{w}} f - \deg_{\mathbf{w}} f - \deg_{\mathbf{w}} g + \deg_{\mathbf{w}} df \wedge dg.$$

Now, let us prove Theorem 1.10. Let k_0 be the field of fractions of k . Then, we may regard F as an element of $S(\mathbf{w}, k_0)$. Hence, in proving (i), we

may assume that k is a field by replacing k with k_0 if necessary. Similarly, since $T_3(k)$ is regarded as a subset of $T_3(k_0)$, we may assume that k is a field in proving (ii). In both (i) and (ii), we may also assume that $f_3 = x_3$ for the following reason. Since k is a field, we can define $H \in T_3(k)$ by $H = (x_1, x_2, f_3)$. Put $G = H^{-1}$. Then, we have $\text{mdeg}_{\mathbf{w}} G = \mathbf{w}$ by Theorem 3.3 (iii), since $\text{mdeg}_{\mathbf{w}} H = \mathbf{w}$. By Theorem 3.3 (ii) and Corollary 2.3 (i), this implies that $\deg_{\mathbf{w}} G(f) = \deg_{\mathbf{w}_G} f$ for each $f \in k[\mathbf{x}]$. Since $\mathbf{w}_G = \text{mdeg}_{\mathbf{w}} G = \mathbf{w}$, it follows that $\deg_{\mathbf{w}} G(f) = \deg_{\mathbf{w}} f$ for each $f \in k[\mathbf{x}]$. Thus, we get $G \circ F \sim_{\mathbf{w}} F$. Therefore, by replacing F with $G \circ F$ if necessary, we may assume that $f_3 = x_3$.

First, we show (i). It suffices to construct $G \in E_3^{\mathbf{w}}(\kappa)$ such that $g_3 = x_3$ and $G \sim_{\mathbf{w}} F$. Assume that f_1 or f_2 belongs to $k[x_i, x_j]$ for some $1 \leq i < j \leq 3$. Since both cases are similar, we only consider the case of f_1 . If $(i, j) = (1, 2)$, then the assertion follows from Proposition 7.2 (ii). Assume that $(i, j) \neq (1, 2)$. Then, we have $j = 3$. Since f_1 belongs to $k[x_i, x_3]$ and $f_3 = x_3$, we may write $f_1 = \alpha_1 x_i + p_1$ and $f_2 = \alpha_2 x_l + p_2$, where $\alpha_1, \alpha_2 \in k^\times$, $p_1 \in k[x_3]$, $p_2 \in k[x_i, x_3]$ and $l \in \{1, 2\}$ with $l \neq i$. Define $p'_1 \in \kappa[x_3]$ by $p'_1 = 0$ if $p_1 = 0$, and $p'_1 = x_3^d$ if $d := \deg p_1 \geq 0$, and $p'_2 \in \kappa[x_i, x_3]$ by $p'_2 = 0$ if $p_2 = 0$, and $p'_2 = x_i^{u_i} x_3^{u_3}$ if $p_2 \neq 0$, where $u_i, u_3 \in \mathbf{N}_0$ are such that $\deg_{\mathbf{w}} p_2 = u_i w_i + u_3 w_3$. Then, $G := (x_i + p'_1, x_l + p'_2, x_3)$ is an element of $E_3^{\mathbf{w}}(\kappa)$ such that $G \sim_{\mathbf{w}} F$.

Assume that f_1 and f_2 do not belong to $k[x_i, x_j]$ for any $1 \leq i < j \leq 3$. Then, $d_i = \deg_{\mathbf{w}} f_i$ is not less than $\max\{w_1, w_2, w_3\}$ for $i = 1, 2$. By assumption, d_i belongs to $\sum_{j \neq i} \mathbf{N}_0 d_j$ for some $1 \leq i \leq 3$. If $i = 3$, then it follows that $d_l \leq d_3$ for $l = 1$ or $l = 2$. Since both cases are similar, we assume that $l = 1$. Then, we have $\max\{w_1, w_2, w_3\} \leq d_1 \leq d_3 = w_3$, and so $d_1 = w_3$. Hence, we may write $f_1 = \alpha_1 x_3 + p_1$, where $\alpha_1 \in k^\times$, and $p_1 \in k[x_1, x_2]$ is such that $\deg_{\mathbf{w}} p_1 \leq w_3$. Then, we have $k[f_1, f_2, f_3] = k[p_1, f_2, f_3]$ since $f_3 = x_3$. By Proposition 7.2 (ii), there exists $G' = (g_1, g_2, x_3) \in E_3^{\mathbf{w}}(\kappa)$ such that $G' \sim_{\mathbf{w}} (p_1, f_2, f_3)$. Then, we have $\deg_{\mathbf{w}} g_1 = \deg_{\mathbf{w}} p_1 \leq w_3 = d_1$. Define $G \in E_3(\kappa)$ by $G = G'$ if $\deg_{\mathbf{w}} g_1 = d_1$, and by $G = (g_1 + x_3, g_2, x_3)$ if $\deg_{\mathbf{w}} g_1 < d_1$. Then, G is an element of $E_3^{\mathbf{w}}(\kappa)$ such that $G \sim_{\mathbf{w}} F$. Next, assume that $i = 1$ or $i = 2$. Since both cases are similar, we assume that $i = 1$. Write $d_1 = l_2 d_2 + l_3 d_3 = l_2 d_2 + l_3 w_3$, where $l_2, l_3 \in \mathbf{N}_0$. Recall that $\deg_{\mathbf{w}} F > |\mathbf{w}|$ by the definition of $S(\mathbf{w}, k)$. Hence, (b) of Theorem 1.1 (i) holds for $I = J = \{1, 2, 3\}$. Since $(f_3^{\mathbf{w}})^{\mathbf{v}} = x_3$ is divisible by x_3 for $\mathbf{v} = 0$, it follows that $I_0 \cap \{1, 2\} \neq \emptyset$. Hence, there exists $s \in \{1, 2\}$ such that d_2 belongs to $\sum_{l \neq s} \mathbf{N}_0 w_l$. Write $d_2 = a w_r + b w_3$, where $a, b \in \mathbf{N}_0$ and $r \in \{1, 2\} \setminus \{s\}$. Since d_1 and d_2 are at least $\max\{w_1, w_2, w_3\}$, we have $d_1 \geq w_r$ and $d_2 \geq w_s$.

Define $G \in E_3(\kappa)$ by

$$G = (x_r + \alpha(x_s + \beta x_r^a x_3^b)^{l_2} x_3^{l_3}, x_s + \beta x_r^a x_3^b, x_3),$$

where $\alpha = 1$ if $d_1 > w_r$, and $\alpha = 0$ if $d_1 = w_r$, and where $\beta = 1$ if $d_2 > w_s$, and $\beta = 0$ if $d_2 = w_s$. Then, G is an element of $E_3^{\mathbf{w}}(\kappa)$ such that $G \sim_{\mathbf{w}} F$. This completes the proof of (i).

Finally, we show (ii). By Theorem 1.9, F admits an elementary reduction for the weight \mathbf{w} . Hence, we have $\deg_{\mathbf{w}}(f_i - h) < \deg_{\mathbf{w}} f_i$ for some $1 \leq i \leq 3$ and $h \in k[f_{i_1}, f_{i_2}]$, where $i_1, i_2 \in \{1, 2, 3\} \setminus \{i\}$ are such that $i_1 < i_2$. Then, $f_i^{\mathbf{w}}$ belongs to $k[f_{i_1}, f_{i_2}]^{\mathbf{w}}$, since $f_i^{\mathbf{w}} = h^{\mathbf{w}}$. If $f_i^{\mathbf{w}}$ belongs to $k[f_{i_1}^{\mathbf{w}}, f_{i_2}^{\mathbf{w}}]$, then d_i belongs to $\mathbf{N}_0 d_{i_1} + \mathbf{N}_0 d_{i_2}$. Assume that $f_i^{\mathbf{w}}$ does not belong to $k[f_{i_1}^{\mathbf{w}}, f_{i_2}^{\mathbf{w}}]$. Then, we have $k[f_{i_1}, f_{i_2}]^{\mathbf{w}} \neq k[f_{i_1}^{\mathbf{w}}, f_{i_2}^{\mathbf{w}}]$. Hence, $f_{i_1}^{\mathbf{w}}$ and $f_{i_2}^{\mathbf{w}}$ are algebraically dependent over k by Corollary 2.3 (iii). If $i \neq 3$, then we have $i_2 = 3$. Since $f_3 = x_3$, it follows that $f_{i_1}^{\mathbf{w}}$ belongs to $k[f_{i_2}^{\mathbf{w}}] = k[f_3^{\mathbf{w}}] = k[x_3]$. Hence, d_{i_1} belongs to $\mathbf{N}_0 d_{i_2}$. Assume that $i = 3$. Then, there exists $h \in k[f_1, f_2]$ such that $h^{\mathbf{w}} = f_3^{\mathbf{w}}$. Since $f_3^{\mathbf{w}}$ does not belong to $k[f_1^{\mathbf{w}}, f_2^{\mathbf{w}}]$ by assumption, $\deg_{\mathbf{w}}^S h > \deg_{\mathbf{w}} h$ holds for $S = \{f_1, f_2\}$ as remarked before Lemma 9.3. By Lemma 9.3, there exist $l_1, l_2 \in \mathbf{N}$ with $\gcd(l_1, l_2) = 1$ such that $(f_2^{\mathbf{w}})^{l_1} \approx (f_1^{\mathbf{w}})^{l_2}$ and

$$w_3 = \deg_{\mathbf{w}} h \geq l_2 d_1 - d_1 - d_2 + \deg_{\mathbf{w}} df_1 \wedge df_2 > (l_1 l_2 - l_1 - l_2) \frac{1}{l_1} d_1,$$

where the last inequality is because $d_2 = (l_2/l_1)d_1$ and $\deg_{\mathbf{w}} df_1 \wedge df_2 > 0$. Assume that $f_1^{\mathbf{w}}$ or $f_2^{\mathbf{w}}$ does not belong to $k[x_1, x_2]$. Then, we have $\deg_{x_3} f_j^{\mathbf{w}} = l_j d$ for $j = 1, 2$ for some $d \in \mathbf{N}$, since $l_1 \deg_{x_3} f_2^{\mathbf{w}} = l_2 \deg_{x_3} f_1^{\mathbf{w}}$ and $\gcd(l_1, l_2) = 1$. Hence, we get $d_1 = \deg_{\mathbf{w}} f_1 \geq l_1 d w_3 \geq l_1 w_3$. By the preceding inequality, it follows that $l_1 l_2 - l_1 - l_2 < 1$. Since $\gcd(l_1, l_2) = 1$, this implies that $l_1 = 1$ or $l_2 = 1$. Thus, we know that $f_2^{\mathbf{w}} \approx (f_1^{\mathbf{w}})^{l_2}$ or $(f_2^{\mathbf{w}})^{l_1} \approx f_1^{\mathbf{w}}$. Therefore, d_2 belongs to $\mathbf{N}_0 d_1$ or d_1 belongs to $\mathbf{N}_0 d_2$. If $f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ belong to $k[x_1, x_2]$, then the conditions (a) through (d) before Theorem 6.2 are fulfilled, since $\deg_{\mathbf{w}} F > |\mathbf{w}|$ and $f_3 = x_3$. Hence, we have $f_1^{\mathbf{w}} \approx (f_2^{\mathbf{w}})^u$ or $f_2^{\mathbf{w}} \approx (f_1^{\mathbf{w}})^u$ for some $u \geq 1$ by Theorem 6.2 (i). Therefore, d_1 belongs to $\mathbf{N}_0 d_2$ or d_2 belongs to $\mathbf{N}_0 d_1$. This completes the proof of (ii).

To conclude this paper, we mention Takurou Kanehira's master's thesis [8], where he generalized Karaś-Zygadło [9, Theorem 2.1] by means of the generalized Shestakov-Umirbaev theory as follows (cf. [5, Theorem 3.1]; see also [14] for further generalizations, and [25] and [16] for related results).

Theorem 9.4 (Kanehira). *Assume that k is a field of characteristic zero. Let $d_3 \geq d_2 > d_1 \geq 3$ be integers such that d_1 and d_2 are mutually prime*

odd numbers. If there exist $\mathbf{w} \in \mathbf{N}^3$ and $F \in T_3(k)$ such that $\text{mdeg}_{\mathbf{w}} F = (d_1, d_2, d_3)$ and $\deg_{\mathbf{w}} F > |\mathbf{w}|$, then d_3 belongs to $\mathbf{N}_0 d_1 + \mathbf{N}_0 d_2$.

Because of this result, Kanehira studied the following problem and gave some partial results.

Problem 9.5 (Kanehira). Assume that k is a field of characteristic zero. Find sufficient conditions on $\mathbf{w} \in \mathbf{N}^3$ under which the following statement holds: (d_1, d_2, d_3) belongs to $\text{mdeg}_{\mathbf{w}} T_3(k)$ for any mutually prime odd numbers

$$d_1, d_2 \in \bigcup_{1 \leq i < j \leq 3} (w_i \mathbf{N}_0 + w_j \mathbf{N}_0),$$

and $d_3 \in \mathbf{N}_0 d_1 + \mathbf{N}_0 d_2$ such that $3 \leq d_1 < d_2 \leq d_3$ and $d_1 + d_2 + d_3 > |\mathbf{w}|$.

The results presented in this paper may be applicable to such a problem.

References

- [1] T. Asanuma, On strongly invariant coefficient rings, Osaka J. Math. **11** (1974), 587–593.
- [2] S. M. Bhatwadekar and A. K. Dutta, On residual variables and stably polynomial algebras, Comm. Algebra **21** (1993), 635–645.
- [3] H. Derksen, O. Hadas and L. Makar-Limanov, Newton polytopes of invariants of additive group actions, J. Pure Appl. Algebra **156** (2001), no. 2-3, 187–197.
- [4] W. Dicks, Automorphisms of the polynomial ring in two variables, Publ. Sec. Mat. Univ. Autònoma Barcelona **27** (1983), 155–162.
- [5] E. Edo, T. Kanehira, M. Karaś and S. Kuroda, Separability of wild automorphisms of a polynomial ring, Transform. Groups **18** (2013), 81–96.
- [6] H. Jung, Über ganze birationale Transformationen der Ebene, J. Reine Angew. Math. **184** (1942), 161–174.
- [7] M. Karaś, There is no tame automorphism of \mathbb{C}^3 with multidegree $(3, 4, 5)$, Proc. Amer. Math. Soc. **139** (2011), 769–775.
- [8] T. Kanehira, Weighted multidegrees of tame automorphisms of a polynomial ring in three variables (Japanese), Master’s Thesis, Tokyo Metropolitan University, February, 2012.

- [9] M. Karaś and J. Zygałło, On multidegrees of tame and wild automorphisms of \mathbb{C}^3 , J. Pure Appl. Algebra **215** (2011), 2843–2846.
- [10] W. van der Kulk, On polynomial rings in two variables, Nieuw Arch. Wisk. (3) **1** (1953), 33–41.
- [11] S. Kuroda, The infiniteness of the SAGBI bases for certain invariant rings, Osaka J. Math. **39** (2002), 665–680.
- [12] S. Kuroda, A generalization of the Shestakov-Umirbaev inequality, J. Math. Soc. Japan **60** (2008), 495–510.
- [13] S. Kuroda, Shestakov-Umirbaev reductions and Nagata’s conjecture on a polynomial automorphism, Tohoku Math. J. **62** (2010), 75–115.
- [14] S. Kuroda, On the Karaś type theorems for the multidegrees of polynomial automorphisms, arXiv:math.AC/1303.3703v1.
- [15] S. Kuroda, Initial forms of stable invariants for additive group actions, arXiv:math.AC/1304.0313.
- [16] J. Li and X. Du, Multidegrees of tame automorphisms with one prime number, arXiv:math.AC/1204.0930
- [17] L. Makar-Limanov, On Automorphisms of Certain Algebras (Russian), PhD Thesis, Moscow, 1970.
- [18] L. Makar-Limanov, P. van Rossum, V. Shpilrain and J.-T. Yu, The stable equivalence and cancellation problems, Comment. Math. Helv. **79** (2004), 341–349.
- [19] H. Matsumura, Commutative ring theory, translated from the Japanese by M. Reid, second edition, Cambridge Studies in Advanced Mathematics, 8, Cambridge Univ. Press, Cambridge, 1989.
- [20] V. Shpilrain and J.-T. Yu, Affine varieties with equivalent cylinders, J. Algebra **251** (2002), no. 1, 295–307.
- [21] M. Nagata, On Automorphism Group of $k[x, y]$, Lectures in Mathematics, Department of Mathematics, Kyoto University, Vol. 5, Kinokuniya Book-Store Co. Ltd., Tokyo, 1972.
- [22] T. Oda, *Convex bodies and algebraic geometry*, translated from the Japanese, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 15, Springer, Berlin, 1988.

- [23] I. Shestakov and U. Umirbaev, Poisson brackets and two-generated subalgebras of rings of polynomials, *J. Amer. Math. Soc.* **17** (2003), 181–196.
- [24] I. Shestakov and U. Umirbaev, The tame and the wild automorphisms of polynomial rings in three variables, *J. Amer. Math. Soc.* **17** (2003), 197–227.
- [25] X. Sun and Y. Chen, Multidegrees of Tame Automorphisms in Dimension Three, *Publ. Res. Inst. Math. Sci.* **48** (2012), 129–137.

Department of Mathematics and Information Sciences
Tokyo Metropolitan University
1-1 Minami-Osawa, Hachioji
Tokyo 192-0397, Japan
kuroda@tmu.ac.jp